1. Determine whether the following series converge or diverge. Show all work to justify your answers.

   (a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 - 5}{n^4 - n^2 + 1}$

   Solution: Using Limit Comparison Test with the convergent p-series $\sum \frac{1}{n^2}$:

   $$\lim_{n\to\infty} \frac{3n^2 - 5}{n^4 - n^2 + 1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{3n^4 - 5n^2}{n^4 - n^2 + 1} = 3 < \infty$$

   Thus by Limit Comparison Test, the given series also $\boxed{\text{converges}}$

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}$

   Solution: This series $\boxed{\text{diverges}}$ by divergence test, since

   $$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = \exp\left(\lim_{n\to\infty} \frac{\ln n}{n}\right)$$
   $$\overset{L'H}{=} \exp\left(\lim_{n\to\infty} \frac{1}{n}\right) = e^0 = 1 \neq 0$$

2. Determine whether the following series converge or diverge. Show all work to justify your answers.

   (a) (8 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

   Solution: We can use the integral test, since $\dfrac{1}{x\ln x}$ is continuous, positive, and decreasing on the interval $x > 2$. Using $u = \ln x$,

   $$\int_2^{\infty} \frac{1}{x\ln x}\, \mathrm{d}x = \int \frac{1}{u}\, \mathrm{d}u = \ln|u| = \ln|\ln x|\Big|_2^{\infty} = \infty$$

   Thus by the integral test, the series $\boxed{\text{diverges}}$

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

   Solution: By the root test:

   $$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{n+1}\right)^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

   the series $\boxed{\text{converges}}$

3. (a) (5 points) Show the following series converges

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+5}$$

Solution: To use the Alternating Series Test, we check that $b_n = \frac{1}{3n+5}$ satisfies the conditions:

- Decreasing ($b_{n+1} < b_n$): $\frac{1}{3(n+1)+5} < \frac{1}{3n+5}$ ✓
- $b_n \to 0$: $\lim_{n\to\infty} \frac{1}{3n+5} = 0$ ✓

Thus by the Alternating Series Test, this series converges.

(b) (6 points) Find the minimum M that guarantees

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+5} - \sum_{n=1}^{M} \frac{(-1)^n}{3n+5} \right| < 0.01$$

Solution:

$$\begin{aligned} \frac{1}{3(M+1)+5} < \frac{1}{100} &\implies 100 < 3(M+1)+5 \\ &\implies \frac{95}{3} < M+1 \\ &\implies M > \frac{92}{3} = 30.\overline{6} \end{aligned}$$

So $\boxed{M = 31}$ suffices.

4. Determine whether the following series converge conditionally, converge absolutely, or diverge. Justify your answer.

(a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n)}{2^n}$

Solution: Since $\dfrac{\left|\cos(n)\right|}{2^n} < \dfrac{1}{2^n}$ and $\sum \dfrac{1}{2^n}$ converges (geometric series with $r = \frac{1}{2} < 1$), $\sum \dfrac{|\cos(n)|}{2^n}$ converges by direct comparison test. Thus the given series $\boxed{\text{converges absolutely}}$

(b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Solution: Note that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

so

$$\lim_{n\to\infty} \frac{(-1)^n n}{n+1} \quad \text{DNE}$$

Thus by the divergence test, the series $\boxed{\text{diverges}}$

5. Determine for which values of x the following power series converge.

(a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5x^2}{(n+5)n!}$

Solution:

$$\lim_{n\to\infty}\left|\frac{5x^{n+1}}{(n+6)(n+1)!}\cdot\frac{(n+5)n!}{5x^n}\right| = \lim_{n\to\infty}\left|x\frac{n+5}{(n+6)(n+1)}\right|$$
$$= |x|\lim_{n\to\infty}\frac{n+5}{(n+6)(n+1)} = 0$$

Thus the power series converges for all $\boxed{x \in \mathbb{R}}$

(b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x-1)^n}{3^n}$

Solution:

$$\lim_{n\to\infty}\left|\frac{(n+1)^2(x-1)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n^2(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\cdot\frac{x-1}{3}\right|$$
$$= \frac{1}{3}|x-1|\lim_{n\to\infty}\frac{(n+1)^2}{n^2}$$
$$= \frac{1}{3}|x-1| < 1 \implies |x-1| < 3$$

At $x=4$: $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges

At $x=-2$: $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^2$ diverges

Thus the power series converges for $\boxed{x \in (-2,4)}$

6. (a) (8 points) Find the power series for the function $f(x) = \dfrac{2}{1+x^3}$. Determine the interval of convergence of the series.

Solution:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

$$\frac{2}{1+x^3} = \frac{2}{1-(-x^3)} = 2\sum_{n=0}^{\infty}(-x^3)^n \qquad (|-x^3| < 1 \implies |x| < 1)$$

$$= 2\sum_{n=0}^{\infty}(-1)^n x^{3n} \qquad (|x| < 1)$$

The interval of convergence of this series is $\boxed{(-1, 1)}$

(b) (8 points) Consider the power series $g(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$. Find the power series for $\displaystyle\int \frac{g(x)}{x}\,\mathrm{d}x$.

Solution:

$$\frac{g(x)}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$$

$$\int \frac{g(x)}{x}\,\mathrm{d}x = \boxed{\sum_{n=1}^{\infty} \frac{x^n}{n^2}}$$

7. (9 points) Find the degree three Taylor polynomial of

$$f(x) = \ln(1-x)$$

centered at $x = 0$.

Solution:

$$\begin{aligned} f(x) &= \ln(1-x) & f(0) &= 0 \\ f'(x) &= -\frac{1}{1-x} & f'(0) &= -1 \\ f''(x) &= -(1-x)^{-2} & f''(0) &= -1 \\ f'''(x) &= -2(1-x)^{-3} & f'''(0) &= -2 \end{aligned}$$

so

$$T_3(x) = 0 + (-1)x + (\frac{-1}{2!})x^2 + (\frac{-2}{3!})x^3 = \boxed{-x - \frac{1}{2}x^2 - \frac{1}{3}x^3}$$