1. Evaluate the following integrals.

(a) (10 points) $\displaystyle\int \frac{e^x}{(1+e^x)^3}\,\mathrm{d}x$

Solution: Let $u = e^x$, so $\mathrm{d}u = e^x\,\mathrm{d}x$. Then

$$\frac{e^x}{(1+e^x)^3}\,\mathrm{d}x = \int u^{-3}\,\mathrm{d}u = \frac{u^{-2}}{-2} = \boxed{-\frac{1}{2}(1+e^x)^{-2} + C}$$

(b) (10 points) $\displaystyle\int x\sqrt{x-1}\,\mathrm{d}x$

Solution: Letting $u = x-1$, so $\mathrm{d}u = \mathrm{d}x$ and $u+1 = x$ Then

$$\begin{aligned}
\int x\sqrt{x-1}\,\mathrm{d}x &= \int (u+1)\sqrt{u}\,\mathrm{d}u \\
&= \int u^{3/2} + u^{1/2}\,\mathrm{d}u \\
&= \frac{2}{5}u^{5/2} + \frac{3}{2}u^{3/2} \\
&= \boxed{\frac{2}{5}(x-1)^{5/2} + \frac{3}{2}(x-1)^{3/2} + C}
\end{aligned}$$

2. Evaluate the following integrals.

(a) (10 points) $\displaystyle\int x^2 \ln(x)\,\mathrm{d}x$

Solution: IBP:

$$\begin{array}{ccc} & D & I \\ + & \ln x & x^2 \\ - & \dfrac{1}{x} & \dfrac{x^3}{3} \end{array}$$

so

$$\begin{aligned}\int x^2 \ln(x)\,\mathrm{d}x &= \frac{x^3}{3}\ln x - \int \frac{1}{3}x^2\,\mathrm{d}x \\ &= \boxed{\frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C}\end{aligned}$$

(b) (10 points) $\displaystyle\int \tan^{-1} x\,\mathrm{d}x$, where $\tan^{-1} x = \arctan x$.

Solution: IBP:

$$\begin{array}{ccc} & D & I \\ + & \tan^{-1} x & 1 \\ - & \dfrac{1}{x^2+1} & x \end{array}$$

so

$$\begin{aligned}\int \tan^{-1} x\,\mathrm{d}x &= x\tan^{-1} x - \int \frac{x}{x^2+1}\,\mathrm{d}x \\ &= \boxed{x\tan^{-1} x - \frac{1}{2}\ln\left(x^2+1\right) + C}\end{aligned}$$

3. Evaluate the following integrals.

(a) (10 points) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{4-x^2}}$

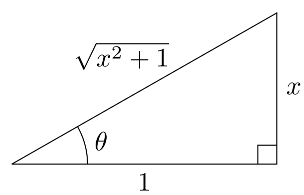
Solution: Using the general formula $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$ on the cover page:

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{4-x^2}} &= \sin^{-1}\left(\frac{x}{2}\right)\Bigg|_0^1 \\ &= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) \\ &= \frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}\end{aligned}$$

(b) (10 points) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$

Solution: Using trig sub $x = \tan\theta$, so $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$,

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sqrt{1+\tan^2\theta}} \\ &= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sqrt{\sec^2\theta}} \\ &= \int \sec\theta\,\mathrm{d}\theta \\ &= \ln|\sec\theta + \tan\theta| \\ &= \boxed{\ln\left|\sqrt{x^2+1}+x\right| + C}\end{aligned}$$



4. Evaluate the following integrals.

(a) (10 points) $\displaystyle\int \sin^3(x)\cos^8(x)\,\mathrm{d}x$

Solution:

$$\begin{aligned}
\sin^3(x)\cos^8(x)\,\mathrm{d}x &= \int \sin x\,\underbrace{\left(1-\cos^2(x)\right)}_{=\sin^2(x)}\cos^8(x)\,\mathrm{d}x \\
&= -\int (1-u^2)u^8\,\mathrm{d}u \qquad (u=\cos x;\, \mathrm{d}u = -\sin x\,\mathrm{d}x) \\
&= -\int (u^8-u^{10})\,\mathrm{d}u \\
&= -\frac{1}{9}u^9 + \frac{1}{11}u^{11} \\
&= \boxed{-\frac{1}{9}\cos^9(x) + \frac{1}{11}\cos^{11}(x) + C}
\end{aligned}$$

(b) (10 points) $\displaystyle\int \tan^4(x)\,\mathrm{d}x$

Solution: Using reduction formula on cover page:

$$\begin{aligned}
\int \tan^4(x)\,\mathrm{d}x &= \frac{\tan^3(x)}{3} - \int \tan^2(x)\,\mathrm{d}x \\
&= \frac{\tan^3(x)}{3} - \left(\tan x - \int 1\,\mathrm{d}x\right) \\
&= \boxed{\frac{\tan^3(x)}{3} - \tan x + x + C}
\end{aligned}$$

5. (10 points) An object moves along a straight line with velocity function $v(t) = te^{-t}$, in meters per second. Determine its change in position over the time interval $t = 0$ to $t = 4$ seconds.

Solution: The object's displacement is given by the integral $\int_0^4 te^{-t}\,\mathrm{d}t$. Evaluating this uses IBP:

$$\begin{array}{ccc} & D & I \\ + & t & e^{-t} \\ - & 1 & -e^{-t} \\ + & 0 & e^{-t} \end{array}$$

Thus

$$\begin{aligned} \int_0^4 te^{-t}\,\mathrm{d}t &= -te^{-t} - e^{-t}\Big|_0^4 \\ &= -4e^{-4} - e^{-4} - (0-1) = \boxed{-5e^{-4}+1} \end{aligned}$$

6. (10 points) Find a function $f(t)$ such that $f'(t) = s\tan\big(s^2\big) - \sec^2(s)$.

Solution:

$$\int \Big(s\tan\big(s^2\big) - \sec^2(s)\Big)\,\mathrm{d}t = \boxed{-\frac{1}{2}\ln\Big|\cos\big(s^2\big)\Big| - \tan(s) + C}$$

(Integrating the first term involves the u-sub $u = s^2$.)