1. Evaluate the following integrals.

(a) (10 points) $\displaystyle\int \frac{3x^2+2x+13}{x^2+4}\,\mathrm{d}x$

Solution: Long division gives

$$\begin{array}{r|l} & 3 \\ \hline x^2+4 & 3x^2+2x+13 \\ & -3x^2 \qquad -12 \\ \hline & 2x \quad +1 \end{array}$$

so

$$\begin{aligned}\int \frac{3x^2+2x+13}{x^2+4}\,\mathrm{d}x &= \int \left(3+\frac{2x+1}{x^2+4}\right)\mathrm{d}x \\ &= 3x + \int \frac{2x}{x^2+4}\,\mathrm{d}x + \int \frac{1}{x^2+4}\,\mathrm{d}x \\ &= \boxed{3x+\ln\left(x^2+4\right)+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C}\end{aligned}$$

(b) (12 points) $\displaystyle\int \frac{2x^2-5x-1}{x^3-x}\,\mathrm{d}x$

Solution: Partial Fractions:

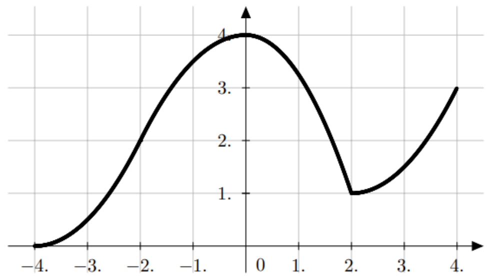
$$\frac{2x^2-5x-1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

Clearing denominators:

$$2x^2-5x-1 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

Solving gives $A=1$, $B=-2$, $C=3$. Hence

$$\begin{aligned}\int \frac{2x^2-5x-1}{x^3-x}\,\mathrm{d}x &= \int \left(\frac{1}{x}+\frac{-2}{x-1}+\frac{3}{x+1}\right) \\ &= \boxed{\ln|x|-2\ln|x-1|+3\ln|x+1|+C}\end{aligned}$$



2. For the function $y = f(x)$ graphed above, approximate the definite integral $\int_{-4}^{4} f(x)\,\mathrm{d}x$ using

   (a) (6 points) The Trapezoidal rule for T_4.

   Solution: $\Delta x = \frac{4-(-4)}{4} = 2$

   $$\begin{aligned} T_4 &= \frac{\Delta x}{2}\big(f(-4) + 2f(-2) + 2f(0) + 2f(2) + f(4)\big) \\ &= 1\big(0 + 2(2) + 2(4) + 2(1) + 3\big) = \boxed{17} \end{aligned}$$

   (b) (6 points) Simpson's rule for S_4.

   Solution: Again, $\Delta x = \frac{4-(-4)}{4} = 2$.

   $$\begin{aligned} S_4 &= \frac{\Delta x}{3}\big(f(-4) + 4f(-2) + 2f(0) + 4f(2) + f(4)\big) \\ &= \frac{2}{3}\big(0 + 4(2) + 2(4) + 4(1) + 3\big) = \boxed{\frac{46}{3}} \end{aligned}$$

3. (6 points) Give the partial fraction expansion of the rational function using coefficients $A, B, C, \ldots$, but **do not** solve the values of the coefficients.

$$f(x) = \frac{2x^4 - 5x + 3}{(x^2 + 2x + 1)(x^2 + x + 1)^2}$$

Solution: Notice that $x^2 + 2x + 1 = (x + 1)^2$, whereas $x^2 + x + 1$ is irreducible, so the partial fraction expansion is of the form

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{(x^2 + x + 1)^2}$$

4. Evaluate the integrals using proper limit notation.

(a) (8 points) $\displaystyle\int_0^\infty xe^{-x}\,\mathrm{d}x$

Solution:

$$\lim_{b\to\infty}\int_0^b xe^{-x}\,\mathrm{d}x$$

	D	I
$+$	x	e^{-x}
$-$	1	$-e^{-x}$
$+$	0	e^{-x}

$$\begin{aligned}
&= \lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b\\
&= \lim_{b\to\infty}\left[-be^{-b}-e^{-b}-(-1)\right]\\
&= \lim_{b\to\infty}\frac{-b-1}{e^b}+1\\
&\overset{LH}{=} \lim_{b\to\infty}\frac{-1}{e^b}+1=\boxed{1}
\end{aligned}$$

(b) (8 points) $\displaystyle\int_2^6 \frac{1}{\sqrt{6-x}}\,\mathrm{d}x$

Solution:

$$\begin{aligned}
\lim_{b\to6^-}\int_2^b \frac{1}{\sqrt{6-x}}\,\mathrm{d}x &= \lim_{b\to6^-}\left[-2\sqrt{6-x}\Big|_2^b\right]\\
&= -2\lim_{b\to6^-}\left(\sqrt{6-b}-\sqrt{4}\right)\\
&= -2(0-2)=\boxed{4}
\end{aligned}$$

5. (8 points) A spring requires a force of 2 newtons to stretch it 1 meter beyond its rest length. How much work is required to stretch the spring from 1 meter to 3 meters beyond its rest length?

Solution: The first sentence allows us to compute the spring constant:

$$F=kx \implies 2=k(1)\implies k=2$$

Then

$$W=\int_1^3 2x\,\mathrm{d}x = x^2\Big|_1^3 = 9-1=\boxed{8\text{ Joules}}$$

6. (a) (8 points) Find the arc length of the curve $y = x^3$, $0 \leq x \leq 1$. Just set up the integral. **Do not evaluate.**

Solution:

$$L = \int_0^1 \sqrt{1 + (3x^2)^2}\,\mathrm{d}x = \boxed{\int_0^1 \sqrt{1 + 9x^4}\,\mathrm{d}x}$$

(b) (10 points) Find the surface area of the surface generated by rotating the curve in part (a) around the x-axis. **Evaluate the integral.**

Solution:

$$\begin{aligned} SA &= \int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\,\mathrm{d}x \qquad (u = 1 + 9x^4; \mathrm{d}u = 36x^3\,\mathrm{d}x) \\ &= \frac{\pi}{18} \int_1^{10} \sqrt{u}\,\mathrm{d}u \\ &= \frac{\pi}{18} \cdot \frac{2}{3} \left[u^{3/2}\right]_1^{10} \\ &= \boxed{\frac{\pi}{27}(10^{3/2} - 1)} \end{aligned}$$

7. (10 points) A cylindrical tank of radius 5 ft and height 10 ft is filled with water of density ρ lb/ft^3. How much work is required to pump all of the water out of the top of the tank? Set up and evaluate an appropriate integral.

Solution: Let x denote the distance to the top of the tank. The force to move a slice of water of width Δx is $F = ma = \rho V = \rho\pi r^2 h = \rho\pi 25\Delta x$. Work is then

$$W = \int_0^{10} \rho 25\pi x \,\mathrm{d}x = \rho 25\pi \int_0^{10} x \,\mathrm{d}x = \rho 25\pi \left[\frac{x^2}{2}\right]_0^{10} = \boxed{\rho 1250\pi \text{ ft lbs}}$$

8. (10 points) Find the centroid $(\overline{x}, \overline{y})$ of the region bounded by the semicircle $y = \sqrt{9 - x^2}$, $-3 \le x \le 3$ and the x-axis. (You may use symmetry and the area formula for a circle.)

Solution: By symmetry of the region, $\overline{x} = 0$. To compute $\overline{y}$:

$$m = \int_{-3}^{3} \sqrt{9 - x^2} = \frac{1}{2}\pi \cdot 9 = \frac{9}{2}\pi \qquad \text{(formula for area of semicircle)}$$

$$M_x = \frac{1}{2}\int_{-3}^{3} (9 - x^2)\,\mathrm{d}x \qquad \text{(symmetry of even functions)}$$

$$= \frac{1}{2} \cdot 2 \int_0^3 (9 - x^2)\,\mathrm{d}x$$

$$= 9x - \frac{x^3}{3}\bigg|_0^3 = (27 - 9) - 0 = 18$$

$$\overline{y} = \frac{M_x}{m} = 18 \cdot \frac{2}{9\pi} = \frac{4}{\pi}$$

Thus the centroid is $\boxed{(0, \frac{4}{\pi})}$.