1. Evaluate the following.

   (a) (8 points) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, e^{2x}\sinh^{-1}(\sqrt{x})$, where $\sinh^{-1}$ is the inverse sinh function.

   Solution:

   $$2e^{2x}\sinh^{-1}(\sqrt{x}) + e^{2x}\frac{1}{\sqrt{1+x}}\cdot\frac{1}{2\sqrt{x}}$$

   (b) (8 points) $\displaystyle\int \sqrt{1+x^2}\,\mathrm{d}x$, using the substitution $x = \sinh(\theta)$.

   Solution:

   $$\begin{aligned}
   &\int \sqrt{1+x^2}\,\mathrm{d}x \\
   &\quad= \int \sqrt{1+\sinh^2(\theta)}\cosh(\theta)\,\mathrm{d}\theta \\
   &\quad= \int \sqrt{\cosh^2(\theta)}\cosh(\theta)\,\mathrm{d}\theta \\
   &\quad= \int \cosh^2(\theta)\,\mathrm{d}\theta && \left(\cosh^2\theta = \frac{1}{2}(1+\cosh(2\theta))\right) \\
   &\quad= \int \frac{1}{2}(1+\cosh(2\theta))\,\mathrm{d}\theta \\
   &\quad= \frac{1}{2}\theta + \frac{1}{4}\sinh(2\theta) && (\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)) \\
   &\quad= \frac{1}{2}\theta + \frac{1}{2}\sinh(\theta)\cosh(\theta) && (\cosh^2\theta - \sinh^2\theta = 1) \\
   &\quad= \boxed{\frac{1}{2}\sinh^{-1}(x) + \frac{1}{2}x\sqrt{1+x^2} + C}
   \end{aligned}$$

   Remark: Since $x = \sinh(\theta)$, we can use the hyperbolic pythagorean identity to get $\cosh\theta = \sqrt{1+\sinh^2\theta} = \sqrt{1+x^2}$.

2. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln(x)}{xy^2}, \qquad (x > 0).$$

(a) (8 points) Find the general solution.

Solution:

$$\int y^2\,\mathrm{d}y = \int \frac{\ln(x)}{x}\,\mathrm{d}x$$
$$\implies \frac{1}{3}y^3 = \frac{1}{2}(\ln(x))^2 + C_1$$
$$\implies y^3 = \frac{3}{2}(\ln(x))^2 + C_2$$
$$\implies \boxed{y(x) = \sqrt[3]{\frac{3}{2}(\ln(x))^2 + C}}$$

(b) (2 points) Find the solution satisfying the initial condition $y(1) = 4$.

Solution:

$$y(1) = \sqrt[3]{\frac{3}{2}\cdot 0 + C} = 4 \implies C = 4^3 = 64$$
$$\boxed{y(x) = \sqrt[3]{\frac{3}{2}(\ln(x))^2 + 64}}$$

3. Find the limit of the sequence or state that it diverges.

(a) (6 points) $\lim_{n\to\infty} \frac{(\ln n)^2}{n}$

Solution:

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} \overset{LH}{=} \lim_{n\to\infty} \frac{2\ln n \cdot \frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} \overset{LH}{=} \lim_{n\to\infty} \frac{2 \cdot \frac{1}{n}}{1} = \boxed{0}$$

(b) (6 points) $\lim_{n\to\infty} n \sin(2/n)$

Solution: This has type $\infty \cdot 0$.

$$\lim_{n\to\infty} n\sin(2/n) = \lim_{n\to\infty} \frac{\sin(2/n)}{n^{-1}} \overset{LH}{=} \lim_{n\to\infty} \frac{\cos(2/n) \cdot -2n^{-2}}{-n^{-2}}$$
$$= \lim_{n\to\infty} 2\cos(2/n) = \boxed{2}$$

4. Evaluate the series.

(a) (6 points) $\sum_{n=2}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$

Solution: Geometric series with $r = -\frac{2}{3}$, and initial term $a = (-1)^2 \left(\frac{2}{3}\right)^2 = \frac{4}{9}$. The sum is thus

$$\frac{a}{1-r} = \frac{\frac{4}{9}}{1-(-\frac{2}{3})} = \frac{4}{9} \cdot \frac{3}{5} = \boxed{\frac{4}{15}}$$

(b) (8 points) $\sum_{n=3}^{\infty} \frac{2}{n^2-1}$

Solution:

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$$
$$\sum_{n=3}^{\infty} \frac{2}{n^2-1} = \sum_{n=3}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$
$$= \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots$$
$$= \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

5. (7 points) Use the limit comparison test to determine whether the following series converges or diverges. Show all work to justify your answer. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+8}{4n^4-n^2}$

Solution: Compare against $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\dfrac{n^2+8}{4n^4-n^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+8n^2}{4n^4-n^2} = \frac{1}{4}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test. So by Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+8}{4n^4-n^2}$ $\boxed{\text{converges}}$

6. Determine whether the following series converge or diverge. Show all work to justify your answers.

(a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Solution: $\boxed{\text{Converges}}$ by p-series test, with $p = 3/2 > 1$.

(b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \cos(1/n)$

Solution: $\boxed{\text{Diverges}}$ by divergence test since $\lim_{n\to\infty} \cos(1/n) = 1 \neq 0$.

(c) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Solution: $\boxed{\text{Diverges}}$ by integral test, since $f(x) = \dfrac{1}{x\ln x}$ is a positive, continuous, decreasing function on $(2,\infty)$ and

$$\int_2^{\infty} \frac{1}{x\ln x}\,\mathrm{d}x = \ln|\ln x|\,\Big|_2^{\infty} \qquad \text{diverges}$$

7. (6 points) Find the minimum M that guarantees that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} - \sum_{n=1}^{M} \frac{(-1)^{n+1}}{2n-1} \right| < 0.01.$$

Solution: The expression in absolute values is bounded above by the $(M+1)$-st term in the series. Thus we want

$$\begin{aligned} \frac{1}{2(M+1)-1} < 0.01 = \frac{1}{100} &\implies 100 < 2(M+1) - 1 \\ &\implies \frac{101}{2} - 1 < M \\ &\implies M > 49.5 \end{aligned}$$

So $\boxed{M = 50}$ works.

8. Determine whether the following series converge conditionally, converge absolutely, or diverge. Justify your answer.

   (a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series test ($p = \frac{1}{2} < 1$).

$\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\displaystyle\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ for $n \geq 1$, so by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\boxed{\text{converges conditionally}}$

   (b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n)}{5e^n}$

Solution: Since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}\sin(n)}{5e^n} \right| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{5e^n} \leq \sum_{n=1}^{\infty} \frac{1}{5e^n}$$

and the right series converges, being a geometric series with $r = \frac{1}{e} \approx \frac{1}{2.7} < 1$, by the Direct Comparison Test, the original series $\boxed{\text{converges absolutely}}$