1. (10 points) Evaluate the series $\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n$

Solution: This is a geometric series. It converges to the value

$$\frac{(\frac{4}{5})^2}{1-\frac{4}{5}} = \frac{16}{25}\cdot 5 = \boxed{\frac{16}{5}}$$

2. (10 points) Find the minimum M that guarantees that

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{10n+2} - \sum_{n=1}^{M}\frac{(-1)^{n+1}}{10n+2} - \right| < .001.$$

Solution:

$$\begin{aligned} \left|\frac{(-1)^{M+2}}{10(M+1)+2}\right| &< \frac{1}{1000} \\ \implies 10(M+1)+2 &> 1000 \\ \implies M+1 &> \frac{998}{10} = 99.8 \\ \implies M &> 98.8 \end{aligned}$$

Therefore $\boxed{M = 99}$ suffices.

3. (10 points) Determine the radius of convergence and interval of convergence, but don't check the endpoints
$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$.

Solution:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}(4x-8)^{n+1}}{\frac{2^n}{n}(4x-8)^n}\right| \\ &= \lim_{n\to\infty}\left|2(4x-8)\frac{n}{n+1}\right| \\ &= 2|4x-8| < 1 \\ &\implies 8|x-2| < 1 \implies |x-2| < \frac{1}{8}\end{aligned}$$

The radius of convergence is $\boxed{R = \frac{1}{8}}$

The interval of convergence is $\boxed{\left(\frac{15}{8}, \frac{17}{8}\right)}$

4. (10 points) Show whether the series converges absolutely or conditionally or diverges. Name all tests used.
$\sum_{n=6}^{\infty}(-1)^n \frac{4+\sqrt{n}}{n^3-5n^2}$

Solution: Using Limit Comparison Test with $\sum \frac{1}{n^{5/2}}$ (which converges by p-series test with $p = \frac{5}{2} > 1$):

$$\lim_{n\to\infty}\left|\frac{4+\sqrt{n}}{n^3-5n}\cdot n^{5/2}\right| = \lim_{n\to\infty}\left|\frac{n^3+4n^{5/2}}{n^3-5n}\right| = 1$$

By Limit Comparison Test, the given series $\boxed{\text{converges absolutely}}$

5. (10 points) Find the Taylor Series for $f(x) = e^{-x}$ about $x = -4$.

Solution:

$$\begin{array}{ll} f(x) = e^{-x} & f(-4) = e^{4} \\ f'(x) = -e^{-x} & f'(-4) = -e^{4} \\ f''(x) = e^{-x} & f''(-4) = e^{4} \end{array}$$

The Taylor series is thus

$$e^4 - e^4(x+4) + \frac{e^4}{2!}(x+4)^2 - \frac{e^4}{3!}(x+4)^3 \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n e^4}{n!}(x+4)^n}$$

6. (10 points) Find the limit of the sequence or state that it diverges. $\lim_{n\to\infty} n^2 e^{-n}$

Solution:

$$\lim_{n\to\infty} n^2 e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{e^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{e^n} = \boxed{0}$$

Determine whether the following series converge conditionally, converge absolutely, or diverge. Name all tests used.

7. (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{5/3}}$

Solution: $\displaystyle\sum \frac{1}{n^{5/3}}$ converges by p-series test ($p = \frac{5}{3} > 1$), so the given series $\boxed{\text{converges absolutely}}$

8. (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Solution: The absolute value version of the series $\displaystyle\sum \frac{1}{\sqrt{n}}$ diverges by p-series test ($p = \frac{1}{2} < 1$).

Trying the Alternating Series Test:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad \text{and} \quad \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

so $\displaystyle\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test.

Thus the given series $\boxed{\text{converges conditionally}}$

9. (10 points) Find a power series representation for the following function and determine its interval of convergence.
$f(x) = \frac{x}{5-x}$

Solution: One answer is:

$$\frac{x}{5-x} = \frac{x-5+5}{5-x} = -1 + \frac{5}{5-x} = -1 + \frac{5}{1-(x-4)}$$
$$= \boxed{-1 + 5\sum_{n=0}^{\infty}(x-4)^n} \qquad (|x-4| < 1)$$

which has interval of convergence $\boxed{(3,5)}$

Another answer is:

$$\frac{x}{5-x} = x \cdot \frac{1}{5-x} = \frac{x}{5} \cdot \frac{1}{1-\frac{x}{5}} = \frac{x}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \qquad \left(\left|\frac{x}{5}\right| < 1 \implies |x| < 5\right)$$
$$= \boxed{\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^{n+1}} \qquad (|x| < 5)$$

which has interval of convergence $\boxed{(-5,5)}$

10. (10 points) Determine whether the following series converge conditionally, converge absolutely, or diverge. Name all tests used.
$\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^n$

Solution: Using root test:

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{3n+1}{4-2n}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{3n+1}{4-2n}\right| = \frac{3}{2} > 1$$

Thus by root test, the series $\boxed{\text{diverges}}$

11. (10 points) Solve the differential equation for initial conditions $y(1) = 4$ (Not required to solve for y).
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2y - 3x^2$

Solution:

$$
\begin{aligned}
&= x^2(y-3)\\
\int \frac{1}{y-3}\,\mathrm{d}y &= \int x^2\,\mathrm{d}x\\
\ln|y-3| &= \frac{x^3}{3} + C_1\\
|y-3| &= e^{\frac{x^3}{3}+C_1} = e^{\frac{x^3}{3}}e^{C_1} = C_2e^{\frac{x^3}{3}}\\
y-3 &= C_3e^{\frac{x^3}{3}}\\
y(x) &= Ce^{\frac{x^3}{3}} + 3
\end{aligned}
$$

Plugging in the initial condition:

$$4 = Ce^{\frac{1}{3}} + 3 \implies 1 = Ce^{\frac{1}{3}} \implies C = e^{-\frac{1}{3}}$$

The solution is thus

$$\boxed{y(x) = e^{-\frac{1}{3}}e^{\frac{x^3}{3}} + 3}$$