1. Evaluate the following integrals.

(a) (10 points) $\displaystyle\int \frac{x^3+2x+1}{x^2+4}\,\mathrm{d}x$

Solution: Long division gives

$$
\begin{array}{r@{}l}
 & \quad\quad\quad\; x \\
x^2+4 & \overline{)\; x^3+2x+1} \\
 & \;\; -x^3-4x \\
 & \;\;\overline{\quad\quad -2x+1}
\end{array}
$$

so

$$
\begin{aligned}
\int \frac{x^3+2x+1}{x^2+4}\,\mathrm{d}x &= \int \left(x + \frac{-2x+1}{x^2+4}\right)\mathrm{d}x \\
&= \frac{1}{2}x^2 - \int \frac{2x}{x^2+4}\,\mathrm{d}x + \int \frac{1}{x^2+4}\,\mathrm{d}x \\
&= \boxed{\frac{1}{2}x^2 - \ln\bigl(x^2+4\bigr) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}
\end{aligned}
$$

(b) (12 points) $\displaystyle\int \frac{3x+5}{(x^2+2x+1)(x+2)}\,\mathrm{d}x$

Solution: Partial Fractions:

$$\frac{3x+5}{(x+1)^2(x+2)} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Clearing denominators:

$$3x+5 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$$

Solving gives $A=-1$, $B=1$, $C=2$. Hence

$$
\begin{aligned}
\int \frac{3x+5}{(x^2+2x+1)(x+2)}\,\mathrm{d}x &= \int \left(\frac{-1}{x+2} + \frac{1}{x+1} + \frac{2}{(x+1)^2}\right) \\
&= \boxed{-\ln|x+2| + \ln|x+1| - \frac{2}{x+1} + C}
\end{aligned}
$$

2. Approximate the definite integral $\int_{-4}^{4} \sqrt{16 - x^2}\,\mathrm{d}x$ using

(a) (8 points) The Midpoint rule for M_4. (Do not simplify the arithmetic.)

Solution: $\Delta x = \frac{4-(-4)}{4} = \frac{8}{4} = 2$. The values are $x_i = -4, -2, 0, 2, 4$. The midpoints are $m_i = -3, -1, 1, 3$. Midpoint rule says

$$\begin{aligned} M_4 &= \Delta x \left(f(-3) + f(-1) + f(1) + f(3)\right) \\ &= \boxed{2 \cdot \left(\sqrt{7} + \sqrt{15} + \sqrt{15} + \sqrt{7}\right)} \end{aligned}$$

(b) (8 points) Simpson's rule for S_4. (Do not simplify the arithmetic.)

Solution: $\Delta x = \frac{4-(-4)}{4} = \frac{8}{4} = 2$. The values are $x_i = -4, -2, 0, 2, 4$. Simpson's rule says

$$\begin{aligned} S_4 &= \frac{\Delta x}{3} \left(f(-4) + 4f(-2) + 2f(0) + 4f(2) + f(4)\right) \\ &= \boxed{\frac{2}{3}\left(0 + 4\sqrt{12} + 2\sqrt{16} + 4\sqrt{12} + 0\right)} \\ &= \frac{2}{3}\left(0 + 8\sqrt{3} + 8 + 8\sqrt{3} + 0\right) \end{aligned}$$

3. (8 points) A spring requires a force of 4 newtons to stretch 2 meters beyond its rest length. How much work is required to stretch the spring from 2 meters to 4 meters beyond its rest length?

Solution: The first sentence allows us to determine the spring constant k:

$$F = kx \implies 4 = k(2) \implies k = 2$$

Then

$$W = \int_2^4 2x\,\mathrm{d}x = x^2\Big|_2^4 = 16 - 4 = \boxed{12 \text{ Joules}}$$

4. Evaluate the following improper integrals, or state that they do not exist. Use proper limit notation.

(a) (6 points) $\displaystyle\int_2^5 \frac{\mathrm{d}x}{\sqrt{x-2}}$

Solution:

$$\begin{aligned}\int_2^5 \frac{\mathrm{d}x}{\sqrt{x-2}} = \lim_{b\to 2^-}\int_b^5 \frac{\mathrm{d}x}{\sqrt{x-2}} &= \lim_{b\to 2^-}\left[2\sqrt{x-2}\Big|_b^5\right] \\ &= \lim_{b\to 2^-} 2\left(\sqrt{3}-\sqrt{b-2}\right) \\ &= 2\sqrt{3} - 0 = \boxed{2\sqrt{3}}\end{aligned}$$

(b) (6 points) $\displaystyle\int_3^\infty \frac{\mathrm{d}x}{(x-2)^3}$

Solution:

$$\begin{aligned}\int_3^\infty \frac{\mathrm{d}x}{(x-2)^3} = \lim_{b\to\infty}\int_3^b \frac{\mathrm{d}x}{(x-2)^3} &= \lim_{b\to\infty}\left[\frac{(x-2)^{-2}}{-2}\Bigg|_3^b\right] \\ &= -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{(b-2)^2} - \frac{1}{(3-2)^2}\right) \\ &= -\frac{1}{2}(0-1) = \boxed{\frac{1}{2}}\end{aligned}$$

(c) (4 points) $\displaystyle\int_{-2}^2 \frac{\mathrm{d}x}{x^2}$

Solution:

$$\begin{aligned}\int_{-2}^2 \frac{\mathrm{d}x}{x^2} &= \lim_{a\to 0^-}\int_{-2}^a x^{-2}\,\mathrm{d}x + \lim_{b\to 0^+}\int_b^2 x^{-2}\,\mathrm{d}x \\ &= \lim_{a\to 0^-}\left[-x^{-1}\Big|_{-2}^a\right] + \lim_{b\to 0^+}\left[-x^{-1}\Big|_b^2\right] \\ &= -\lim_{a\to 0^-}\left[\frac{1}{a} - \frac{1}{-2}\right] - \lim_{b\to 0^+}\left[\frac{1}{2} - \frac{1}{b}\right] \\ &= -\left(-\infty + \frac{1}{2}\right) - \lim_{b\to 0^+}\left[\frac{1}{2} - \frac{1}{b}\right]\end{aligned}$$

The integral $\boxed{\text{diverges / does not exist}}$

5. (a) (8 points) Find the arc length of the curve $y = \sin x$, $0 \le x \le \frac{\pi}{2}$. Just set up the integral. **Do not evaluate.**

Solution:

$$L = \boxed{\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,\mathrm{d}x}$$

(b) (10 points) Find the surface area of the surface generated by rotating the curve in part (a) around the x-axis. **Evaluate the integral.** Make use of an appropriate integral formula on the cover page.

Solution:

$$\begin{aligned}
SA &= \int_0^{\pi/2} 2\pi \sin x\sqrt{1+\cos^2 x}\,\mathrm{d}x \qquad (u = \cos x; \mathrm{d}u = -\sin x\,\mathrm{d}x) \\
&= -2\pi \int_1^0 \sqrt{1+u^2}\,\mathrm{d}u \\
&= 2\pi \int_0^1 \sqrt{u^2+1}\,\mathrm{d}u \\
&= 2\pi \left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u+\sqrt{u^2+1}\right|\right)\right]_0^1 \qquad \text{(formula on cover page)} \\
&= \pi\left(\sqrt{2} + \ln\left|1+\sqrt{2}\right| - (0 + \ln|1|)\right) \\
&= \boxed{\pi\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)}
\end{aligned}$$

6. (10 points) How much work is done by winding up a hanging cable of length 50 feet and weight density 2 lb/ft.

Solution: Each segment of cable of length Δx has force $2\Delta x$ lbs. A segment at height $x_i \in [0, 50]$ will need to be displaced $50 - x_i$. The total work is thus

$$\begin{aligned} W &= \sum_{x_i} (50 - x_i) 2\Delta x \\ &\xrightarrow{N\to\infty} \int_0^{50} (50 - x) 2 \, \mathrm{d}x \\ &= 100x - x^2 \Big|_0^{50} = \boxed{2500 \text{ ft-lbs}} \end{aligned}$$

7. (10 points) Find the centroid $(\overline{x}, \overline{y})$ of the region bounded by the semicircle $y = \sqrt{4 - x^2}$, $-2 \le x \le 2$ and the x-axis. (You may use the area formula for a circle, and symmetry to determine one of the values $\overline{x}, \overline{y}$.)

Solution: Due to symmetry of the region, we know $\overline{x} = 0$. To compute $\overline{y}$:

$$\begin{aligned} m &= \int_{-2}^{2} \sqrt{4 - x^2} \, \mathrm{d}x = \frac{1}{2}\pi \cdot 4 = 2\pi \qquad \text{(area formula)} \\ M_x &= \frac{1}{2} \int_{-2}^{2} (4 - x^2) \, \mathrm{d}x \qquad \text{(symmetry of even functions)} \\ &= \frac{1}{2} \cdot 2 \int_0^2 (4 - x^2) \, \mathrm{d}x \\ &= 4x - \frac{x^3}{3} \Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \\ \overline{y} &= \frac{M_x}{m} = \frac{16}{3} \cdot \frac{1}{2\pi} = \frac{8}{3\pi} \end{aligned}$$

Thus the centroid is $\boxed{(0, \frac{8}{3\pi})}$