1. Evaluate the following.

   (a) (8 points) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, x^3 \tanh^{-1}(e^{2x})$, where $\tanh^{-1}$ is the inverse tanh function.

   Solution:

   $$3x^2 \tanh^{-1}(e^{2x}) + x^3 \frac{1}{1-(e^{2x})^2} \cdot e^{2x} \cdot 2$$

   (b) (8 points) $\displaystyle\int (2+\cosh x)(1+\sinh x)\,\mathrm{d}x$

   Solution:

   $$\begin{aligned}
   &\int (2+\cosh x)(1+\sinh x)\,\mathrm{d}x \\
   &\quad= \int (2 + 2\sinh x + \cosh x + \sinh x \cosh x)\,\mathrm{d}x \\
   &\quad= 2x + 2\cosh x + \sinh x + \int \sinh x \cosh x\,\mathrm{d}x \qquad \begin{pmatrix} u = \sinh x; \\ \mathrm{d}u = \cosh x\,\mathrm{d}x \end{pmatrix} \\
   &\quad= 2x + 2\cosh x + \sinh x + \int u\,\mathrm{d}u \\
   &\quad= 2x + 2\cosh x + \sinh x + \frac{u^2}{2} \\
   &\quad= \boxed{2x + 2\cosh x + \sinh x + \frac{(\sinh x)^2}{2} + C}
   \end{aligned}$$

   Remark: Alternatively, the u-sub can be performed with $u = \cosh x$, $\mathrm{d}u = \sinh x\,\mathrm{d}x$. The final result will have the term $\frac{(\cosh x)^2}{2}$ instead of $\frac{(\sinh x)^2}{2}$. These are off by a constant since $\cosh^2 x - \sinh^2 x = 1$.

2. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{\ln y}$$

(a) (8 points) Find the general solution. Solve for y explicitly.

Solution:

$$\int \frac{\ln y}{y}\,\mathrm{d}y = \int x\,\mathrm{d}x$$
$$\frac{(\ln y)^2}{2} = \frac{1}{2}x^2 + C_1$$
$$(\ln y)^2 = x^2 + C_2$$
$$\ln y = \pm\sqrt{x^2 + C_2}$$
$$\boxed{y(x) = \exp\left(\pm\sqrt{x^2 + C}\right)}$$

(b) (2 points) Find the solution satisfying the initial condition $y(0) = e$ where e is the natural log base.

Solution:

$$y(0) = \exp\left(\pm\sqrt{0 + C}\right) = e \implies \pm\sqrt{C} = 1 \implies C = 1$$
$$\boxed{y(x) = \exp\left(\sqrt{x^2 + 1}\right)}$$

3. (6 points) Find the limit of the sequence or state that it diverges.

(a) (6 points) $\lim_{n\to\infty} e^{-2n}(n^2 + 1)$

Solution:

$$\lim_{n\to\infty} \frac{n^2 + 1}{e^{2n}} \overset{LH}{=} \lim_{n\to\infty} \frac{2n}{2e^{2n}} \overset{LH}{=} \lim_{n\to\infty} \frac{2}{4e^{2n}} = \boxed{0}$$

(b) (6 points) $\lim_{n\to\infty} \frac{\sin(5/n)}{\sin(3/n)}$

Solution:

$$\lim_{n\to\infty} \frac{\sin(5/n)}{\sin(3/n)} \overset{LH}{=} \lim_{n\to\infty} \frac{\cos(5/n)\cdot -5n^{-2}}{\cos(3/n)\cdot -3n^{-2}} = \lim_{n\to\infty} \frac{5\cos(5/n)}{3\cos(3/n)} = \boxed{\frac{5}{3}}$$

4. Evaluate the series.

(a) (6 points) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2} \cdot \left(\frac{2}{5}\right)^n = \frac{5}{2} + 1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots$

Solution: Geometric series with first term $a = \frac{5}{2}$ and ratio $r = \frac{2}{5}$. Sum is

$$\frac{a}{1-r} = \frac{\frac{5}{2}}{1-\frac{2}{5}} = \frac{5}{2} \cdot \frac{5}{3} = \boxed{\frac{25}{6}}$$

(b) (8 points) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2 - n}$

Solution:

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{2}{n^2-n} &= \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n}\right) \\ &= \left(\frac{2}{1} - \cancel{\frac{2}{2}}\right) + \left(\cancel{\frac{2}{2}} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \frac{2}{4}\right) + \cdots \\ &= \boxed{2}\end{aligned}$$

5. (7 points) Use the limit comparision test to determine whether the following series converges or diverges. Show all work to justify your answer. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n}{3n^{5/2} + 217}$

Solution: Compare against $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{\dfrac{n^2-n}{3n^{5/2}+217}}{\dfrac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^{5/2} - n^{3/2}}{3n^{5/2}+217} = \frac{1}{3}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series test. So by the Limit Comparison Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n}{3n^{5/2}+217}$ $\boxed{\text{diverges}}$

6. Determine whether the following series converge or diverge. Show all work to justify your answers.

(a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution: $\boxed{\text{Diverges}}$ by the p-series test ($p = \frac{1}{2} < 1$)

(b) (7 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

Solution: $\boxed{\text{Converges}}$ by the integral test: $f(x) = \dfrac{1}{x(\ln x)^2}$ is positive, continuous, and decreasing for $x \geq 2$, and

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \int u^{-2}\,\mathrm{d}u = -u^{-1} = -\frac{1}{\ln x}\bigg|_2^{\infty} = -0 + \frac{1}{\ln 2} \qquad \text{converges}$$

(c) (7 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cos\bigl(n\pi/2\bigr)$

Solution: $\boxed{\text{Diverges}}$ by divergence test, since $\displaystyle\lim_{n\to\infty} (-1)^{n+1}\cos\bigl(n\pi/2\bigr)$ does not exist (the sequence cycles between the values 0, 1, 0, and -1)

7. (6 points) The infinite series $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-1}$ is estimated using the M-th partial sum S_M. Find the minimum M that guarantees that $|S - S_M| < 0.01$.

Solution: This is an alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $a_n = \dfrac{1}{3n-1}$. The estimate for error is $|S - S_M| < a_{M+1}$. We want

$$\begin{aligned} \frac{1}{3(M+1)-1} < 0.01 = \frac{1}{100} &\implies 100 < 3(M+1) - 1 \\ &\implies \frac{101}{3} - 1 < M \\ &\implies M > \frac{98}{3} \approx 32.66 \end{aligned}$$

so $\boxed{M = 33}$ works.

8. Determine whether the following series converge conditionally, converge absolutely, or diverge. Justify your answer.

   (a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series).

   $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$ and $\displaystyle\frac{1}{n+1} \leq \frac{1}{n}$ for $n \geq 1$, so by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

   Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\boxed{\text{converges conditionally}}$

   (b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2+n}$

   Solution:
   $$\sum_{n=1}^{\infty} \left|\frac{\sin(2n)}{n^2+n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2+n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$
   The rightmost series converges by p-series test ($p = 2 > 1$).

   Hence by Direct Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(2n)}{n^2+n}\right|$ converges.

   Thus by definition, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2+n}$ $\boxed{\text{converges absolutely}}$