1. (10 points) Evaluate the following integral.
$\displaystyle\int x\tan^{-1}(x)\,\mathrm{d}x$, where $\tan^{-1}(x)$ is the inverse tangent function.

Solution:

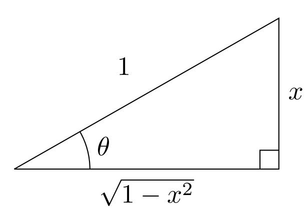
$$\begin{array}{ccc} & D & I \\ + & \tan^{-1}(x) & x \\ - & \dfrac{1}{x^2+1} & \dfrac{x^2}{2} \end{array}$$

$$\begin{aligned}\int x\tan^{-1}(x)\,\mathrm{d}x &= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{x^2+1}\,\mathrm{d}x \\ &= \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \left(1-\frac{1}{x^2+1}\right)\mathrm{d}x \\ &= \boxed{\frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C}\end{aligned}$$

2. (12 points) Evaluate the following integral. $\displaystyle\int \frac{x^2\,\mathrm{d}x}{\sqrt{1-x^2}}$

Solution: $x=\sin\theta$, $\mathrm{d}x=\cos\theta\,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{x^2\,\mathrm{d}x}{\sqrt{1-x^2}} &= \int \frac{\sin^2\theta\cos\theta\,\mathrm{d}\theta}{\sqrt{1-\sin^2\theta}} \\ &= \int \sin^2\theta\,\mathrm{d}\theta \\ &= -\frac{\sin\theta\cos\theta}{2} + \frac{1}{2}\int 1\,\mathrm{d}\theta \\ &= -\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta \\ &= \boxed{-\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C}\end{aligned}$$



3. (10 points) Evaluate the integral. $\displaystyle\int \frac{e^{2x}}{5+e^{2x}}\,\mathrm{d}x$

Solution: $u = e^{2x}$, $\mathrm{d}u = 2e^{2x}\,\mathrm{d}x$ so

$$\begin{aligned}\int \frac{e^{2x}}{5+e^{2x}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{5+u}\,\mathrm{d}u\\ &= \frac{1}{2}\ln|u+5|\\ &= \boxed{\frac{1}{2}\ln\left(e^{2x}+5\right)+C}\end{aligned}$$

4. (12 points) Evaluate the integral. $\displaystyle\int \frac{3x^2+8}{x^3+4x}\,\mathrm{d}x$

Solution:

$$\frac{3x^2+8}{x(x^2+4)} = \frac{2}{x} + \frac{x}{x^2+4}$$

$$\begin{aligned}\int \frac{3x^2+8}{x^3+4x}\,\mathrm{d}x &= \int \left(\frac{2}{x}+\frac{x}{x^2+4}\right)\mathrm{d}x\\ &= \boxed{2\ln|x| + \frac{1}{2}\ln\left(x^2+4\right)+C}\end{aligned}$$

5. (10 points) Evaluate the improper integral or show that it diverges. Use limit notation.
$\displaystyle\int_1^5 \frac{\mathrm{d}x}{\sqrt{x-1}}$

Solution:

$$\begin{aligned}\lim_{b\to 1^-}\int_b^5 \frac{\mathrm{d}x}{\sqrt{x-1}} &= \lim_{b\to 1^-}\left[2\sqrt{x-1}\Big|_b^5\right]\\ &= \lim_{b\to 1^-}\left(2\sqrt{4}-2\sqrt{b-1}\right) = 4-0=\boxed{4}\end{aligned}$$

6. (8 points) Explain why the following series converges, and then evaluate it.

$$\sum_{n=1}^{\infty} \frac{2}{\pi}\left(-\frac{\pi}{4}\right)^n$$

Solution: This is a geometric series with $a = \frac{2}{\pi}(-\frac{\pi}{4}) = -\frac{1}{2}$ and $r = -\frac{\pi}{4}$. It converges since $\left|-\frac{\pi}{4}\right| < 1$. The series converges to

$$\frac{a}{1-r} = \frac{-\frac{1}{2}}{1-(-\frac{\pi}{4})} = \frac{-\frac{1}{2}}{\frac{\pi+4}{4}} = \boxed{\frac{-2}{\pi+4}}$$

7. Let R be the region trapped between $y = x$ and $y = x^2$, with $0 \le x \le 1$.

(a) (6 points) Find the area of the region R.

Solution:

$$\int_0^1 (x - x^2)\,\mathrm{d}x = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \boxed{\frac{1}{6}}$$

(b) (10 points) Find $\overline{y}$, the y coordinate of the centroid of R. (Do not calculate $\overline{x}$.)

Solution:

$$M_x = \frac{1}{2}\int_0^1 \left(x^2 - (x^2)^2\right)\mathrm{d}x = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{15}$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{15}\cdot 6 = \boxed{\frac{2}{5}}$$

8. Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$.

   (a) (4 points) Explain why the series converges.

   Solution: The sequence $b_n = \frac{1}{2n+1}$ is decreasing and has $\lim_{n\to\infty} b_n = 0$, so by Alternating Series Test, the series converges.

   (b) (4 points) How many terms are required to approximate S with an error less than 0.01?

   Solution: The error of an alternating series is bounded by the next term:

$$|S - S_N| \leq b_{N+1}.$$

   So to ensure the error bounds, we enforce

$$\begin{aligned} \frac{1}{2(N+1)+1} \leq 0.01 = \frac{1}{100} &\implies 100 \leq 2(N+1)+1 \\ &\implies \frac{99}{2} - 1 \leq N \\ &\implies N \geq \frac{97}{2} = 48.5 \end{aligned}$$

   Thus $N = 49$ suffices.

9. (12 points) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 4^n}$. (Make clear the status of any end points.)

Solution:

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\cdot 4^{n+1}} \cdot \frac{n \cdot 4^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{x-2}{4} \cdot \frac{n}{n+1} \right| = \frac{1}{4}|x-2| \cdot 1 < 1$$

$$\implies |x-2| < 4$$

$$\implies -2 < x < 6$$

At $x = 6$:

$$\sum_{n=1}^{\infty} \frac{4^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic series)}$$

At $x = -2$:

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges (alternating harmonic series)}$$

Thus the interval of convergence is $\boxed{[-2, 6)}$

10. (12 points) Solve the initial value problem, $(1+t^2)\,\frac{\mathrm{d}y}{\mathrm{d}t} = 2te^y$, $y(0) = 1$. Express your final answer in the form $y = f(t)$.

Solution:

$$\begin{aligned}
(1+t^2)\,\frac{\mathrm{d}y}{\mathrm{d}t} &= 2te^y \\
\int e^{-y}\,\mathrm{d}y &= \int \frac{2t}{1+t^2}\,\mathrm{d}t \\
-e^{-y} &= \ln\big(1+t^2\big) + C \\
y &= -\ln\Big(-\ln\big(1+t^2\big) - C\Big) \\
y(0) &= -\ln\big(-\ln(1) - C\big) = 1 \\
&\implies \ln(-C) = -1 \\
&\implies C = -e^{-1}
\end{aligned}$$

$$\implies \boxed{y(t) = -\ln\Big(-\ln\big(1+t^2\big) + e^{-1}\Big)}$$

11. Determine whether the following series converge or diverge. State clearly which test you are using and implement the test as clearly as you can. The answer is worth 2 points and the work you show 4 points.

(a) (6 points) $\displaystyle\sum_{n=3}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solution: Comparing with the series $\displaystyle\sum \frac{1}{n^{3/2}}$:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Since $\sum \frac{1}{n^{3/2}}$ converges (p-series test with $p = 3/2 > 1$), by Limit Comparison Test, the series $\boxed{\text{converges}}$

Remark: Direct Comparison Test can also be used, since $\displaystyle\sum \frac{\sqrt{n}}{n^2+1} < \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$

(b) (6 points) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(1/n)}{2+\sin(1/n)}$

Solution: Since

$$\lim_{n\to\infty} \frac{\cos(1/n)}{2+\sin(1/n)} = \frac{1}{2} \neq 0$$

this series $\boxed{\text{diverges}}$ by Divergence Test.

12. (6 points) Determine whether the following series converges or diverges. State clearly which test you are using and implement the test as clearly as you can. The answer is worth 2 points and the work you show 4 points.

$$\sum_{n=1}^{\infty} \left(\frac{2n^2-3}{n^2+7n}\right)^n$$

Solution: Using root test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^2-3}{n^2+7n}\right)^n} = \lim_{n\to\infty} \frac{2n^2-3}{n^2+7n} = 2 > 1$$

By root test, the series $\boxed{\text{diverges}}$

13. (12 points) Find the third degree Taylor polynomial $T_3(x)$ for the function $f(x) = \cos x$ centered at $x = \pi/2$.

Solution:

$$\begin{aligned} f(x) &= \cos x & f(\pi/2) &= 0 \\ f'(x) &= -\sin x & f'(\pi/2) &= -1 \\ f''(x) &= -\cos x & f''(\pi/2) &= 0 \\ f'''(x) &= \sin x & f'''(\pi/2) &= 1 \end{aligned}$$

$$\begin{aligned} T_3(x) &= 0 + -1\left(x - \frac{\pi}{2}\right) + \frac{0}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 \\ &= \boxed{-\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3} \end{aligned}$$

14. (8 points) Use the series expansions for e^x and $(1+x)^n$ on the cover page to find the terms up to x^4 for the Maclaurin series of $e^{x^2}\sqrt{1+x^2}$

Solution:

$$\begin{aligned} e^{x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots \\ (1+x^2)^{1/2} &= 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^4 + \cdots \\ &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots \\ e^{x^2}\sqrt{1+x^2} &= \left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots\right) + \left(x^2 + \frac{1}{2}x^4 + \cdots\right) + \left(\frac{1}{2}x^4 + \cdots\right) + \cdots \\ &= \boxed{1 + \frac{3}{2}x^2 + \frac{7}{8}x^4} + \cdots \end{aligned}$$

15. (a) (5 points) Use an appropriate series for the cover sheet to find the Maclaurin series for $\dfrac{1}{1+x^5}$

Solution:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n$$

$$\frac{1}{1+x^5} = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{5n}}$$

(b) (3 points) Use your answer to part (a) to find the Maclaurin series for $\dfrac{x^3}{1+x^5}$

Solution:

$$\frac{x^3}{1+x^5} = x^3 \sum_{n=0}^{\infty} (-1)^n x^{5n} = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{5n+3}}$$

(c) (7 points) Use your answer to part (a) to evaluate the following integral, expressing your final answer as an infinite series. $\displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x^5}$

Solution:

$$\int_0^1 \frac{\mathrm{d}x}{1+x^5} = \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{5n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{5n+1} x^{5n+1} \right]_0^1$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}}$$

16. Consider the curve with parameric equations $x = t^2 + 1$, $y = 2t^3$, $t \geq 0$.

(a) (4 points) Find the slope of the curve at a general value t.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{6t^2}{2t} = \boxed{3t}$$

(b) (4 points) Find the equation of the tangent line to the curve at $t = 1$.

Solution:

$$\begin{aligned} m &= \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1} = 3 \\ x(1) &= 2 \\ y(1) &= 2 \\ &\boxed{y - 2 = 3(x - 2)} \end{aligned}$$

(c) (4 points) Convert the equation to a rectangular equation in x, y.

Solution:

$$\begin{aligned} x &= t^2 + 1 \implies t^2 = x - 1 \\ y &= 2t^3 = \boxed{2(x-1)^{3/2}} \end{aligned}$$

17. (8 points) Find the length of the curve $x = t^2 + 1$, $y = 2t^3$, $0 \le t \le 1$.

Solution:

$$\begin{aligned}
L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t &= \int_0^1 \sqrt{(2t)^2 + (6t^2)^2}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{4t^2 + 36t^4}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{4t^2(1 + 9t^2)}\,\mathrm{d}t \\
&= \int_0^1 2t\sqrt{1 + 9t^2}\,\mathrm{d}t \qquad\qquad (u = 1 + 9t^2; \mathrm{d}u = 18t\,\mathrm{d}t) \\
&= \frac{1}{9}\int_1^{10} \sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{9} \cdot \frac{2}{3}\left[u^{3/2}\right]_1^{10} \\
&= \boxed{\frac{2}{27}(10^{3/2} - 1)}
\end{aligned}$$

18. (a) (6 points) Convert the polar equation $r = 2\sin\theta + 4\cos\theta$, to a rectangular equation in x, y.

Solution:

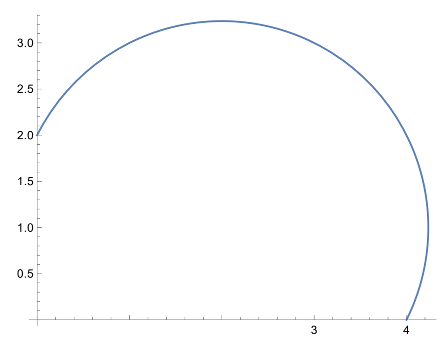
$$r^2 = 2r\sin\theta + 4r\cos\theta$$
$$x^2 + y^2 = 2y + 4x$$
$$x^2 - 4x + 4 + y^2 - 2y + 1 = 4 + 1$$
$$\boxed{(x-2)^2 + (y-1)^2 = (\sqrt{5})^2}$$

(Circle centered at $(2, 1)$ with radius $\sqrt{5}$)

(b) (5 points) Sketch the graph of the polar equation in part (a) with $0 \le \theta \le \frac{\pi}{2}$. (Note: $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$.)

Solution:

θ	$r = 2\sin\theta + 4\cos\theta$
0	$0 + 4 = 4$
$\pi/4$	$2 \cdot \frac{\sqrt{2}}{2} + 4 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2} \approx 4.2$
$\pi/2$	$2 + 0 = 2$



19. (6 points) Find the slope of the polar curve $r = 2\sin\theta + 4\cos\theta$ at $\theta = \pi/2$.

Solution:

$$x = r\cos\theta = (2\sin\theta\cos\theta + 4\cos^2\theta)$$
$$y = r\sin\theta = (2\sin^2\theta + 4\sin\theta\cos\theta)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{2 \cdot 2\sin\theta\cos\theta + 4\sin\theta(-\sin\theta) + 4\cos\theta\cos\theta}{2\sin\theta(-\sin\theta) + 2\cos^2\theta + 8\cos\theta(-\sin\theta)}$$

$$m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/2} = \frac{4 \cdot 0 - 4 + 0}{-2 + 0 + 0} = \frac{-4}{-2} = \boxed{2}$$