Name: Soln

Recitation time: ________ Rec. instructor: ________

MATH 221 - Midterm 1
January 31, 2023

- This exam contains 7 pages (including this cover page) and 7 questions.
- Answer the questions in the spaces provided in this booklet.
- No books, calculators, or notes are allowed. You must show all your work to get credit for your answers.
- You have 1 hour and 15 minutes to complete the exam.

Question:	1	2	3	4	5	6	7	Total
Points:	18	18	18	18	10	8	10	100
Score:								

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \sin^n(x)\ dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\ dx$$

$$\int \cos^n(x)\ dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\ dx$$

$$\int \tan^n(x)\ dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\ dx$$

$$\int \sec^n(x)\ dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\ dx$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

1. Evaluate the following integrals

(a) (9 points) $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int u^{-2}\,du$

$= -u^{-1}$

$= \boxed{-\frac{1}{\ln x} + C}$

(b) (9 points) $\displaystyle\int x^2\sqrt{x^3+5}\,dx$

$u = x^3+5$

$du = 3x^2\,dx$

$= \frac{1}{3}\int \underbrace{\sqrt{u}}_{=u^{\frac{1}{2}}}\,du$

$= \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}$

$= \boxed{\frac{2}{9}(x^3+5)^{\frac{3}{2}} + C}$

2. Evaluate the following integrals.

(a) (9 points) $\int x^5 \ln x \, dx$

	D	I
+	$\ln x$	x^5
−	$\frac{1}{x}$	$\frac{x^6}{6}$

$= \frac{x^6}{6}\ln x - \frac{1}{6}\int x^5 dx$

$= \boxed{\frac{x^6}{6}\ln x - \frac{1}{36}x^6 + C}$

(b) (9 points) $\int \sin^{-1}(x) \, dx$, where $\sin^{-1}(x) = \arcsin(x)$.

	D	I
+	$\sin^{-1}x$	1
−	$\frac{1}{\sqrt{1-x^2}}$	x

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

$u = 1-x^2$
$du = -2x\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int u^{-\frac{1}{2}} du$

$= x\sin^{-1}x + u^{\frac{1}{2}}$

$= \boxed{x\sin^{-1}x + \sqrt{1-x^2} + C}$

3. Evaluate the following integrals.

(a) (9 points) $\displaystyle\int \sec^4(x)\tan^4(x)\,dx$

$\tan^2+1=\sec^2$

$= \int \tan^4 x\,(\tan^2 x+1)\sec^2 x\,dx$

$u=\tan x$
$du=\sec^2 x\,dx$

$= \int u^4(u^2+1)\,du$

$= \int u^6+u^4\,du$

$= \dfrac{u^7}{7}+\dfrac{u^5}{5}$

$= \boxed{\dfrac{\tan^7 x}{7}+\dfrac{\tan^5 x}{5}+C}$

(b) (9 points) $\displaystyle\int \tan^4(x)\,dx$

reduction formula

$= \dfrac{\tan^3 x}{3} - \int \tan^2 x\,dx$

$= \dfrac{\tan^3 x}{3} - \left(\tan x - \int 1\,dx\right)$

$= \boxed{\dfrac{\tan^3 x}{3} - \tan x + x + C}$

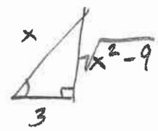
4. Evaluate the following integrals.

$\tan^2+1=\sec^2$

(a) (11 points) $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$



$$= \int \frac{1}{\sqrt{9\sec^2\theta-9}}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{\cancel{3}\,\cancel{\tan\theta}}\cdot \cancel{3}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta|$$

$$= \boxed{\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|+C}$$

(b) (7 points) $\displaystyle\int \frac{5}{16+x^2}\,dx$

Using formula:

$$= 5\int \frac{1}{16+x^2}\,dx$$

$$= \boxed{\frac{5}{4}\tan^{-1}\left(\frac{x}{4}\right)+C}$$

Using u-sub:

$$= 5\int \frac{1}{16\left(1+\left(\frac{x}{4}\right)^2\right)}\,dx$$

$$= \frac{5}{16}\int \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

$u = \frac{x}{4}$

$du = \frac{1}{4}dx$

$$= \frac{5}{4}\int \frac{du}{1+u^2}$$

$$= \frac{5}{4}\tan^{-1}(u)$$

$$= \boxed{\frac{5}{4}\tan^{-1}\left(\frac{x}{4}\right)+C}$$

Using trig sub:

$x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$

$$= 5\int \frac{1}{16(\cancel{1+\tan^2\theta})}\cdot 4\,\cancel{\sec^2\theta}\,d\theta$$

$$= \frac{5}{4}\int 1\,d\theta$$

$$= \frac{5}{4}\theta$$

$$= \boxed{\frac{5}{4}\tan^{-1}\left(\frac{x}{4}\right)+C}$$

5. (10 points) Evaluate the following integral

$$\int_0^{\pi/2} \sin^3(x)\cos^2(x)\,dx$$

$\sin^2 + \cos^2 = 1$

$$= \int_0^{\pi/2} \sin x\,(1-\cos^2 x)\cos^2 x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$x = 0 \to u = 1$
$x = \frac{\pi}{2} \to u = 0$

$$= -\int_1^0 (1-u^2)u^2\,du$$

$$= \int_0^1 u^2 - u^4\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5}\Big]_0^1$$

$$= \frac{1}{3} - \frac{1}{5} = \boxed{\frac{2}{15}}$$

6. (8 points) Find a function $f(t)$ such that

$$f'(t) = \sin(t)\sin(3t)$$

Product-to-sum formula

$$\int \sin t \sin(3t)\,dt$$

$$= \frac{1}{2}\int \cos(-2t) - \cos(4t)\,dt$$

$$= \boxed{\frac{1}{2}\left[-\frac{1}{2}\sin(-2t) - \frac{1}{4}\sin(4t)\right]}$$

7. (10 points) Find the area of the region bounded by the curves $y = 0$, $x = 1$ and $y = xe^{-x}$. $= \frac{x}{e^x}$

$xe^{-x} > 0$ for $x > 0$

$xe^{-x} < 0$ for $x < 0$

$$\int_0^1 xe^{-x}\,dx$$

	D	I
		e^{-x}
+	x	$-e^{-x}$
−	1	e^{-x}
+	0	

$$= -xe^{-x} - e^{-x}$$

$$= (-x-1)e^{-x}\Big]_0^1$$

$$= -2e^{-1} - (\underbrace{-1e^{0}}_{=-1})$$

$$= \boxed{-2e^{-1} + 1}$$