Name: My Soln

Recitation time: ________ Rec. instructor: ________

MATH 221 - Midterm 2
February 28, 2023

- This exam contains 6 pages (including this cover page) and 8 questions.
- Answer the questions in the spaces provided in this booklet.
- No books, calculators, or notes are allowed. You must show all your work to get credit for your answers.
- You have 1 hour and 15 minutes to complete the exam.

Question:	1	2	3	4	5	6	7	8	Total
Points:	11	9	9	11	18	11	11	20	100
Score:									

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C, \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C,$$

$$\int \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\text{Error}(M_n) \le \frac{M(b-a)^3}{24n^2} \quad \text{Error}(T_n) \le \frac{M(b-a)^3}{12n^2} \quad \text{Error}(S_n) \le \frac{M^*(b-a)^5}{180n^4}$$

$$M = \text{max value } |f^{(2)}(x)| \quad M^* = \text{max value } |f^{(4)}(x)|$$

$$M_x = \frac{\rho}{2}\int_a^b f(x)^2 - g(x)^2\, dx \qquad M_y = \rho\int_a^b x(f(x)-g(x))\, dx$$

1. (11 points) Evaluate the following integral

$$\int \frac{3x^4 + x^3 + 13x^2 + 4x + 11}{x^2 + 4}\, dx$$

$$\begin{array}{r l} & 3x^2 + x + 1 \\ x^2+4 \,\big) & 3x^4 + x^3 + 13x^2 + 4x + 11 \\ & -(3x^4 \quad\quad 12x^2) \\ \hline & x^3 + x^2 + 4x \\ & -(x^3 + \quad 4x) \\ \hline & x^2 + 11 \\ & -(x^2 + 4) \\ \hline & 7 \end{array}$$

$= \int 3x^2 + x + 1 + \frac{7}{x^2+4}\, dx$

$= \boxed{x^3 + \frac{x^2}{2} + x + \frac{7}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$

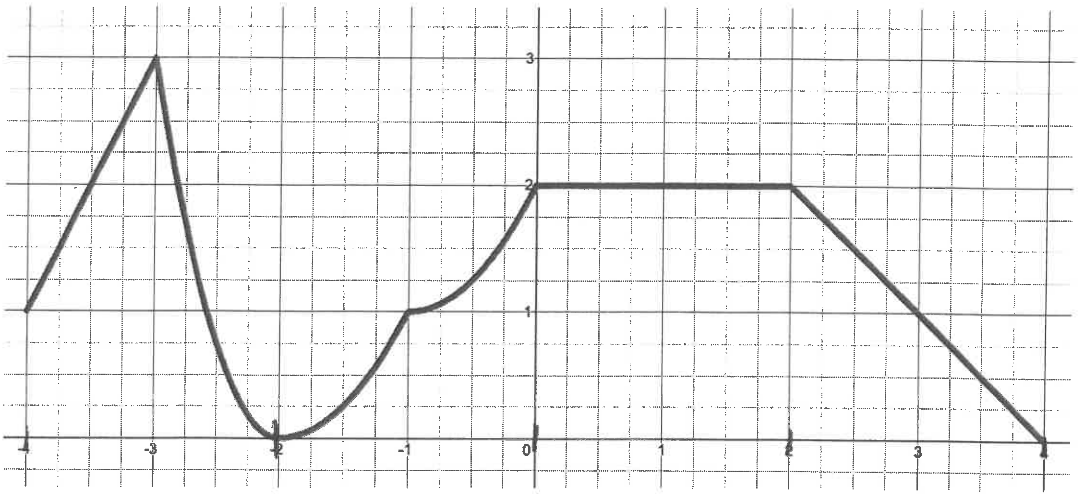
2. (9 points) Give the partial fractions expansion of the rational function $f(x)$ using coefficients $A, B, C, \ldots$, but do NOT solve the values of the coefficients.

$$f(x) = \frac{6x^2 + 3x - 7}{(x^2-1)^2(x^2+4)^2}$$

$(x+1)^2(x-1)^2$

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{(x^2+4)^2}$$

3. (9 points) Consider the function $y = f(x)$ graphed below.



Approximate the definite integral $\int_{-4}^{4} f(x)\,dx$ using the midpoint rule M_4.

$m_i = -3, -1, 1, 3$ $\qquad \Delta x = \frac{4-(-4)}{4} = \frac{8}{4} = 2$

$M_4 = \Delta x\,(f(-3) + f(-1) + f(1) + f(3))$

$= 2(3 + 1 + 2 + 1)$

$= \boxed{14}$

4. (11 points) How large would n need to guarantee that an estimate of $\int_0^2 x^3\,dx$ is accurate to within 0.1 if we use the trapezoid rule T_n ?

$E_{rror}(T_n) \le \frac{M(b-a)^3}{12n^2}$ $\qquad M = \max_{x\in[0,2]} |f''(x)|$

$f' = 3x^2$

$f'' = 6x$

Max at $x = 2$

$M = 12$

$\frac{\cancel{12}(2)^3}{\cancel{12}n^2} \le 0.1 = \frac{1}{10}$

$8 \cdot 10 \le n^2$

$n \ge \sqrt{80} \approx 8.9$ $\qquad \sqrt{81} = 9$

$\boxed{n = 9}$

5. Evaluate the following integrals using proper limit notation

(a) (9 points) $\displaystyle\int_0^{\infty} 3xe^{-x^2}\,dx$

$u = e^{-x^2}$
$du = -2xe^{-x^2}\,dx$

$= \lim\limits_{b\to\infty} \int_0^b 3xe^{-x^2}\,dx$

$= \lim\limits_{b\to\infty} -\frac{3}{2}\int du$

$= \lim\limits_{b\to\infty} -\frac{3}{2}\, e^{-x^2}\Big]_0^b$

$= \lim\limits_{b\to\infty} -\frac{3}{2}\left(e^{-b^2} - \underbrace{e^0}_{=1}\right) = -\frac{3}{2}\Big(\underbrace{\lim\limits_{b\to\infty}\frac{1}{e^{b^2}}}_{=0} - 1\Big) = -\frac{3}{2}(0-1)$

$= \boxed{\frac{3}{2}}$

(b) (9 points) $\displaystyle\int_0^{4} \frac{3}{\sqrt{4-x}}\,dx$

$u = 4-x$
$du = -dx$

$\lim\limits_{b\to 4^-} \int_0^b \frac{3}{\sqrt{4-x}}\,dx$

$= \lim\limits_{b\to 4^-} -3 \int \frac{1}{\sqrt{u}}\,du$ $\quad \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$

$= -3 \lim\limits_{b\to 4^-} 2u^{\frac{1}{2}}$

$= -6 \lim\limits_{b\to 4^-} \sqrt{4-x}\,\Big]_0^b$

$= -6 \lim\limits_{b\to 4^-} \left(\sqrt{4-b} - \sqrt{4}\right)$

$= -6(0 - \underbrace{\sqrt{4}}_{=2}) = -6(-2) = \boxed{12}$

6. (11 points) Find the length of the curve $y = \sqrt{9 - x^2}$, $-3 \le x \le 3$.

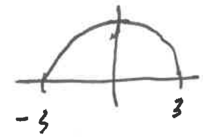
$$L = \int_{-3}^{3} \sqrt{1 + (f')^2}\, dx$$

$$f' = \frac{1}{2\sqrt{9-x^2}} \cdot -2x$$

$$(f')^2 = \frac{x^2}{9-x^2}$$

$$(f')^2 + 1 = \frac{x^2 + 9 - x^2}{9-x^2} = \frac{9}{\cancel{9-x^2}}$$

semicircle of radius 3



$$L = \frac{1}{2} \cdot 2\pi r = \boxed{\pi \cdot 3}$$

$$= \int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\, dx$$

$$= 3 \sin^{-1}\left(\frac{x}{3}\right)\Big]_{-3}^{3}$$

$$= 3\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 3\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{3\pi}$$

7. (11 points) Find the surface area of the surface obtained by rotating the curve $y = x^2$, $1 \le x \le 2$ around the y-axis.

$$SA = \int_{1}^{2} 2\pi x\, ds$$

$$y' = 2x$$
$$(y')^2 = 4x^2$$

$$= 2\pi \int_{1}^{2} x \sqrt{1 + (f')^2}\, dx$$

$$u = 1 + 4x^2$$
$$du = 8x\, dx$$

$$x = 1 \rightarrow u = 5$$
$$x = 2 \rightarrow u = 17$$

$$= 2\pi \int_{1}^{2} x \sqrt{1 + 4x^2}\, dx$$

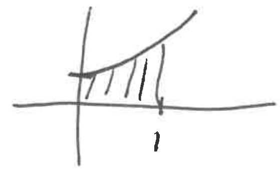
$$= \frac{\pi}{4} \int \sqrt{u}\, du$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big]_{5}^{17}$$

$$= \boxed{\frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)}$$

8. (20 points) Find $\bar{x}$, the x-coordinate of the center of mass (centroid) of the region bounded by $y = e^x$, $x = 0$, $x = 1$ and the x-axis.

$$m = \int_0^1 e^x \, dx = e^x \Big]_0^1 = e - 1$$



$$\bar{x} = \frac{M_y}{m}$$

$$M_y = \int_0^1 x(e^x) \, dx$$

	D	I
+	x	e^x
−	1	e^x
+	0	e^x

$$= xe^x - e^x \Big]_0^1$$

$$= e^x(x-1)\Big]_0^1$$

$$= 0 - 1 \cdot (-1)$$

$$= 1$$

$$\bar{x} = \boxed{\frac{1}{e-1}}$$