Name: My Sol

Recitation time: ______________ Rec. instructor: ______________

MATH 221 - Midterm 3
April 4, 2023

- This exam contains 7 pages (including this cover page) and 6 questions.
- Answer the questions in the spaces provided in this booklet.
- No books, calculators, or notes are allowed. You must show all your work to get credit for your answers.
- You have 1 hour and 15 minutes to complete the exam.

Question:	1	2	3	4	5	6	Total
Points:	16	16	16	18	16	18	100
Score:							

$$\cosh^2(x) - \sinh^2(x) = 1, \quad \cosh^2(x) = \frac{1+\cosh(2x)}{2}, \quad \sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\frac{d}{dx}(\sinh x) = \cosh x, \quad \frac{d}{dx}(\cosh x) = \sinh x, \quad \frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\text{csch}^2 x, \quad \frac{d}{dx}(\text{sech}x) = -\text{sech}x \tanh x, \quad \frac{d}{dx}(\text{csch}x) = -\text{csch}x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}, \quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(\text{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}, \quad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, \quad \frac{d}{dx}(\text{csch}^{-1}x) = \frac{-1}{|x|\sqrt{1+x^2}}$$

1. (a) (8 points) Prove the following identity

$$\sinh^2(x) = \frac{\cosh(2x) - 1}{2}$$

$$LHS = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2\right]$$

$$= \frac{1}{4}\left[e^{2x} - 2 + e^{-2x}\right]$$

$$RHS = \frac{\frac{e^{2x} - e^{-2x}}{2} - 1}{2} = \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x} - \frac{1}{2}$$

(b) (8 points) Calculate the following integral

$$\int \sinh^2(x)\cosh^3(x)\,dx$$

$\cosh^2 - \sinh^2 = 1$

$\cosh^2 = 1 + \sinh^2$

$u = \sinh x$

$du = \cosh x \, dx$

$$= \int \sinh^2 x\,(1 + \sinh^2 x) \cosh x \, dx$$

$$= \int u^2(1 + u^2)\,du$$

$$= \int u^2 + u^4 \, du$$

$$= \frac{u^3}{3} + \frac{u^5}{5}$$

$$= \boxed{\frac{\sinh^3 x}{3} + \frac{\sinh^5 x}{5} + C}$$

2. Consider the differential equation

$$\frac{dy}{dx} = x^4 y^2.$$

(a) (12 points) Find the general solution to the differential equation

Constant soln:

$x^4 y^2 = 0$

$y^2 = 0$

$y = 0$

$$\int y^{-2}\,dy = \int x^4\,dx$$

$$-y^{-1} = \frac{x^5}{5} + C_1$$

$$\frac{1}{y} = \frac{-x^5}{5} + C_2$$

$$y = \frac{1}{\frac{-x^5}{5} + C}$$

General soln:

$$\boxed{y = 0 \quad \text{or} \quad y = \frac{1}{\frac{-x^5}{5} + C}}$$

(b) (4 points) Find the particular solution satisfying $y(0) = 10$.

$$y(0) = \frac{1}{0 + C} = 10$$

$$\Rightarrow \quad C = \frac{1}{10}$$

$$\boxed{y(x) = \frac{1}{\frac{-x^5}{5} + \frac{1}{10}}}$$

3. (a) (8 points) Evaluate the limit of the sequence $\lim_{n} \dfrac{n^2}{2e^n}$. $\frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{n^2}{2e^n} \overset{LH}{=} \lim_{n\to\infty} \frac{2n}{2e^n} \overset{LH}{=} \lim_{n\to\infty} \frac{\cancel{2}}{\cancel{2}e^n} = \boxed{0}$$

$\frac{\infty}{\infty}$

(b) (8 points) Use the squeeze theorem to calculate $\lim_{n} \dfrac{3n - \cos(n)}{2n}$.

$$-1 \le \cos(n) \le 1$$

$$3n-1 \le 3n - \cos(n) \le 3n+1$$

$$\frac{3n-1}{2n} \le \frac{3n-\cos(n)}{2n} \le \frac{3n+1}{2n}$$

$$\lim_{n\to\infty} \frac{3n-1}{2n} = \frac{3}{2}$$

$$\lim_{n\to A} \frac{3n+1}{2n} = \frac{3}{2}$$

$\therefore$ by squeeze thm $\lim_{n\to\infty} \dfrac{3n-\cos(n)}{2n} = \boxed{\dfrac{3}{2}}$

4. Evaluate the series:

(a) (9 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n + 2^n}{7^{n-1}}$

$$= 7 \sum_{n=1}^{\infty} \frac{(-1)^n + 2^n}{7^n} = 7\left[\sum_{n=1}^{\infty} \left(\frac{-1}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n\right]$$

$$= 7\left[\frac{-\frac{1}{7}}{1-(-\frac{1}{7})} + \frac{\frac{2}{7}}{1-\frac{2}{7}}\right]$$

$$= 7\left[\frac{-1}{\cancel{7}} \cdot \frac{\cancel{7}}{8} + \frac{2}{\cancel{7}} \cdot \frac{\cancel{7}}{5}\right]$$

$$= 7\left(-\frac{1}{8} + \frac{2}{5}\right)$$

$$= 7 \cdot \frac{-5+16}{40} = \boxed{\frac{77}{40}}$$

(b) (9 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$. Hint: Use partial fractions.

$$\frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$$

$$= \sum_{n=3}^{\infty} \frac{1}{n-1} - \frac{1}{n}$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots$$

$$= \boxed{\frac{1}{2}}$$

More rigorously:

$$S_k = \sum_{n=3}^{k} = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{k-2}} - \cancel{\frac{1}{k-1}}\right) + \left(\cancel{\frac{1}{k-1}} - \frac{1}{k}\right)$$

$$= \frac{1}{2} - \frac{1}{k}$$

$$\lim_{k\to\infty} S_k = \boxed{\frac{1}{2}}$$

5. Determine whether the following series converge or diverge. Show all work to justify your answers.

(a) (8 points) $\sum_{n=1}^{\infty} \frac{2n-1}{n^4 - n^2 + 7}$

LCT w/ $\sum \frac{1}{n^3}$ (converges p-series $p=3>1$)

$\lim_{n\to\infty} \frac{2n-1}{n^4-n^2+7} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{2n^4-n^3}{n^4-n^2+7} = 2 > 0$

by LCT, given series also $\boxed{\text{converges}}$.

(b) (8 points) $\sum_{n=1}^{\infty} e^{1/n^2}$

$\lim_{n\to\infty} e^{\frac{1}{n^2}} = e^0 = 1 \neq 0$

by divergence test, series $\boxed{\text{diverges}}$

6. Determine whether the following series converge conditionally, converge absolutely, or diverge. Justify your answer.

(a) (9 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$

$\sum_{n=1}^{\infty} \frac{1}{n+5} < \sum_{n=1}^{\infty} \frac{1}{n}$ diverge (p-series, p=1)

$\therefore \sum \frac{1}{n+5}$ diverges by DCT

$b_n = \frac{1}{n+5}$ pos, decr on $n > 1$

$\lim_{n\to\infty} b_n = 0$

by AST, series converges

$\therefore$ $\boxed{\text{converges conditionally}}$

(b) (9 points) $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n)}{n^2}$

$\left| \frac{(-1)^n \sin(n)}{n^2} \right| = \left| \frac{\sin(n)}{n^2} \right| < \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p = 2 > 1$)

by DCT, given series $\boxed{\text{converges absolutely}}$