Name: My Soln

Recitation time: ____________ Rec. instructor: ____________

MATH 221 - Final
May 10, 2023

- This exam contains 11 pages (including this cover page) and 14 questions.
- No books, calculators, or notes are allowed. You must show all your work to get credit for your answers.
- You have 1 hour and 50 minutes to complete the exam.

Question	Points	Score
1	20	
2	20	
3	9	
4	16	
5	12	
6	12	
7	10	
8	10	
9	11	
10	14	
11	12	
12	18	
13	16	
14	20	
Total:	200	

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}b^x = b^x \ln b$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sin^n(x)\ dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\ dx$$

$$\int \cos^n(x)\ dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\ dx$$

$$\int \tan^n(x)\ dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\ dx$$

$$\int \sec^n(x)\ dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\ dx$$

$$M_x = \frac{1}{2}\int_a^b f(x)^2 - g(x)^2 dx \qquad M_y = \int_a^b x(f(x) - g(x))dx$$

$$L = \int_a^b \sqrt{1 + (dy/dx)^2}dx \quad , \quad SA = \int_a^b 2\pi r\sqrt{1 + (dy/dx)^2}dx$$

$$|R_n(x)| \le \frac{K}{(n+1)!}|x - a|^{n+1}, \quad \text{with} \quad K = \max_{a \le c \le x}|f^{(n+1)}(c)|.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad , \quad \ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad , \quad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$A = \int_a^b y(t)x'(t)dt \quad , \quad L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt \quad , \quad SA = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}dt$$

$$A = \frac{1}{2}\int_a^b r^2 d\theta \quad , \quad L = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}d\theta$$

1. Evaluate the following integrals

(a) (10 points) $\int x^2 \ln(x)\, dx$

	D	I
+	$\ln x$	x^2
−	$\frac{1}{x}$	$\frac{x^3}{3}$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$

$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$

(b) (10 points) $\int \frac{3x-2}{x^2-x}\, dx$

$\frac{3x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$= \frac{2}{x} + \frac{1}{x-1}$

$= \int \frac{2}{x} + \frac{1}{x-1} dx$

$= 2 \ln|x| + \ln|x-1| + C$

2. Evaluate the following integrals

(a) (10 points) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

$u = e^x$

$du = e^x\,dx$

$= \displaystyle\int \frac{1}{1+u^2}\,du$

$= \tan^{-1} u$

$= \tan^{-1}(e^x) + C$

(b) (10 points) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$

$= \displaystyle\int \sin(x)\,(1-\cos^2 x)\cos^2 x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$= -\displaystyle\int (1-u^2)\,u^2\,du$

$= -\displaystyle\int u^2 - u^4\,du$

$= -\dfrac{u^3}{3} + \dfrac{u^5}{5}$

$= -\dfrac{\cos^3 x}{3} + \dfrac{\cos^5 x}{5} + C$

3. (9 points) Set up the integral that computes the area of the surface obtained by rotating the curve $y = 1 - x^2$, $-1 \le x \le 1$ around the x–axis. **Do not evaluate the integral.**

$$SA = \int_{-1}^{1} 2\pi y \sqrt{1+(y')^2}\,dx$$

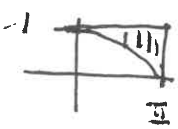
$$y' = -2x$$

$$= \int_{-1}^{1} 2\pi\,(1-x^2)\sqrt{1+4x^2}\,dx$$

4. Let R be the region trapped between $y = 1$ and $y = \cos x$, with $0 \le x \le \frac{\pi}{2}$.

(a) (6 points) Find the area of the region R.

$$\int_0^{\frac{\pi}{2}} 1-\cos x\,dx = x - \sin x\Big]_0^{\frac{\pi}{2}}$$



$$= \frac{\pi}{2} - 1 - (0-0)$$

$$= \boxed{\frac{\pi}{2} - 1}$$

(b) (10 points) Find $\bar{x}$, the x coordinate of the centroid of R. (Do not calculate $\bar{y}$)

$$\bar{x} = \frac{M_y}{M}$$

$$M_y = \int_0^{\pi/2} x(1-\cos x)\,dx$$

	D	I
+	x	$1-\cos x$
−	1	$x - \sin x$
+	0	$\frac{x^2}{2} + \cos x$

$$= x(x - \sin x) - \left(\frac{x^2}{2} + \cos x\right)\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{2} - 1\right) - \frac{\pi^2}{8} - (0 - 0 - 1)$$

$$= \frac{\pi^3}{4} - \frac{\pi}{2} - \frac{\pi^3}{8} + 1$$

$$= \frac{\pi^2}{8} - \frac{\pi}{2} + 1$$

$$\therefore\ \bar{x} = \frac{\frac{\pi^2}{8} - \frac{\pi}{2} + 1}{\frac{\pi}{2} - 1}$$

5. (12 points) Find the general solution of the differential equation

$$\frac{dy}{dx} = (2x+1)y^2$$

$0 = (2x+1)y^2$

$\Rightarrow y^2 = 0$

$\Rightarrow y = 0$

$\int y^{-2}\,dy = \int (2x+1)\,dx$

$-y^{-1} = x^2 + x + C_1$

$y^{-1} = -x^2 - x + C_2$

$\boxed{y = \frac{1}{-x^2-x+C} \quad \text{or} \quad y = 0}$

6. (12 points) Use the integral test to determine if the series $\sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)}$ converges or diverges.

$f(x) = \frac{1}{x \ln^2(x)}$

cont on $x \geq 2$

pos.

decr.

$\therefore$ integral test applies.

$\int_2^{\infty} \frac{1}{x \ln^2 x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int_{\ln 2}^{\infty} u^{-2}\,du$

$= -u^{-1}\Big]_{\ln 2}^{\infty}$

$= 0 + \frac{1}{\ln 2}$

integral converges $\Rightarrow$ series converges by integral test.

7. (10 points) Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{2n-5}{n^4+1}$$

LCT w/ $\sum \underbrace{\frac{1}{n^3}}_{b_n}$

$\lim_{n\to\infty} \frac{2n-5}{n^4+1} \cdot n^3 = \lim_{n\to\infty} \frac{2n^4-5n^3}{n^4+1} = 2 > 0.$ ∴ both converge or diverge

$\sum \frac{1}{n^3}$ is p-series $p=3>1$ ∴ converges

by LCT, given series converges also.

8. (10 points) Evaluate the series $\sum_{n=0}^{\infty} \frac{(-1)^n+4}{3^n}$.

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n + 4\sum_{n=0}^{\infty} \frac{1}{3^n}$$

$$= \frac{1}{1-(-\frac{1}{3})} + 4 \cdot \frac{1}{1-\frac{1}{3}}$$

$$= \frac{1}{1+\frac{1}{3}} + \frac{4}{1-\frac{1}{3}}$$

$$= \frac{3}{3+1} + \frac{12}{3-1}$$

$$= \frac{3}{4} + \frac{12}{2} = \boxed{\frac{27}{4}}$$

9. (11 points) Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges absolutely, conditionally or diverges.

Ratio test: $\frac{a_{n+1}}{a_n}$

$\sum_{n=0}^{\infty} \frac{1}{n!}$ $\quad \rho = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot n!$

$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

∴ converges absolutely by ratio test

∴ original series converges absolutely

(Could just use ratio test on original series)

$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)!} \cdot \frac{(-1)^n (n)!}{} \right| = \lim_{n\to\infty} \left| \frac{-1}{n+1} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$

10. (14 points) Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n}$$

$\rho = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x-2}{3} \cdot \frac{n^2}{(n+1)^2} \right| = \frac{|x-2|}{3} \underbrace{\lim_{n\to\infty} \frac{n^2}{(n+1)^2}}_{=1}$

$= \frac{|x-2|}{3} < 1 \implies |x-2| < 3$

At $x=5$: $\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges p-series $p=2>1$

At $x=-1$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by comparison w/ absolute version.

I.O.C: $[-1, 5]$

11. (12 points) Find the degree two Taylor polynomial of $f(x) = \sqrt{x}$ centered at $x = 1$.

$f(x) = \sqrt{x}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}$ $\qquad f'(1) = \frac{1}{2}$

$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$

$f''(x) = \frac{-1}{4} x^{-\frac{3}{2}}$ $\qquad f''(1) = \frac{-1}{4}$

$$T_2(x) = 1 + \frac{1}{2}(x-1) + \frac{-1}{4}\frac{(x-1)^2}{2!}$$

$$= \boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2}$$

12. Using the appropriate series from the formula sheet, find the Maclaurin series of:

(a) (9 points) $f(x) = \dfrac{x}{1+x^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{x}{1+x^2} = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n+1}}$$

(b) (9 points) $g(x) = \displaystyle\int e^{-x^2}\, dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int e^{-x^2} dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}} \boxed{+C}$$

13. Consider the curve with parametric equations $x = 4 - \sin(2t)$, $y = 5 + \cos(2t)$ for $0 \leq t \leq \pi$.

(a) (5 points) Find the slope of the curve at a general value of t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin(2t)}{-2\cos(2t)} = \boxed{\tan(2t)}$$

(b) (5 points) Find the equation of the tangent line to the curve at $t = \pi/2$.

$x(\frac{\pi}{2}) = 4 - 0 = 4$ $\qquad (4, 4)$

$y(\frac{\pi}{2}) = 5 \cancel{\cos} -1 = 4$

$m = \left.\frac{dy}{dx}\right|_{\frac{\pi}{2}} = \tan(\pi) = \frac{0}{-1} = 0$ $\qquad y - 4 = 0(x-4)$

$\boxed{y = 4}$

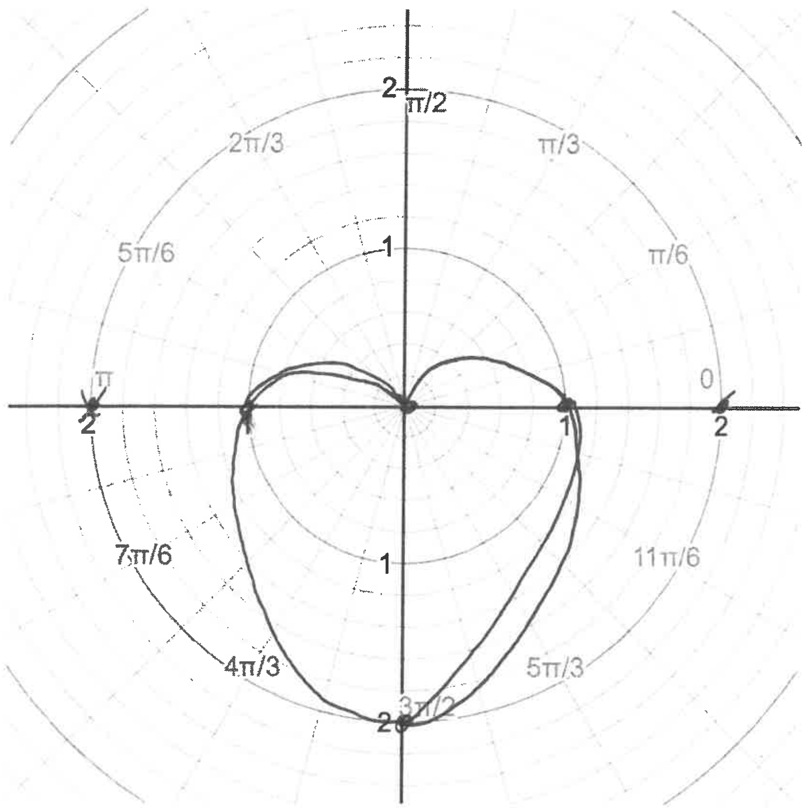
(c) (6 points) Set up the integral that calculates the length of the curve. (Do not evaluate)

$x' = -2\cos(2t)$

$L = \int \sqrt{(x')^2 + (y')^2}\, dt$

$= \boxed{\int_0^{\pi} \sqrt{4\cos^2(2t) + 4\sin^2(2t)}\ dt}$

14. (a) (10 points) Sketch the graph of the polar curve $r = 1 - \sin\theta$.



θ	r
0	$1-0$
$\frac{\pi}{2}$	$1-1=0$
π	$1-0=1$
$\frac{3\pi}{2}$	$1-(-1)=2$

(b) (10 points) Calculate the area bounded by the polar curve $r = 1 - \sin\theta$.

$$A = \frac{1}{2}\int_0^{2\pi} (1-\sin\theta)^2 \, d\theta \qquad (1-\sin\theta)^2 = 1 + \sin^2\theta - 2\sin\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 - \cancel{2\sin\theta}^{\,0} + \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\left[2\pi + \int_0^{2\pi} \sin^2\theta \, d\theta\right]$$

$$= \pi + \frac{1}{2}\left(-\frac{\sin x\cos x}{2} + \frac{1}{2}x\right]_0^{2\pi}$$

$$= \pi + \frac{1}{2}\left(-\cancel{\frac{1}{2}}\,0 + \pi - \cancel{(0+0)}\right)$$

$$= \pi + \frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$$