1. Find the gradient of the following functions:

(a) $f(x, y) = x^2 \sin(xy^2)$.

$$\nabla f(x,y) = \left(2x\sin(xy^2) + x^2y^2\cos(xy^2),\ 2x^3y\cos(xy^2)\right)$$

(b) $g(x, y) = x \sin(x^2y + ye^{3x+4y})$.

$$\nabla g(x,y) = \Big(\sin(x^2y+ye^{3x+4y}) + (2x^2y + 3xye^{3x+4y})\cos(x^2y+ye^{3x+4y}),$$

$$x\left(x^2 + e^{3x+4y} + 4ye^{3x+4y}\right)\cos\left(x^2y+ye^{3x+4y}\right)\Big)$$

2. A certain differentiable function satisfies:

   (a) $f(2,5) = 12$, and $f(3,-4) = -6$.

   (b) $\nabla f(2,5) = (6,8)$, and $\nabla f(3,-4) = (-5,12)$.

   At each of the two points in question (i.e. at $(2,5)$ and at $(3,-4)$) answer the following questions:

   (a) In what direction is the function increasing the fastest and what is the rate of change in that direction?

$\left(\frac{3}{5}, \frac{4}{5}\right)$ & 10 and $\left(\frac{-5}{13}, \frac{12}{13}\right)$ & 13

   (b) What is the directional derivative in the direction of < 3,4 >? (Note: just to be completely clear about semantics here, you are supposed to give the same directional derivative at each point. I did not ask for the directional derivative in the direction of the point $(3,4)$.)

$(6,8)\cdot\left(\frac{3}{5}, \frac{4}{5}\right) = 10$ $(-5,12)\cdot\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{33}{5}$

   (c) What is the tangent plane and/or the linear approximation at each of the two points?

$z = L(x,y) = 12 + 6(x-2) + 8(y-5)$

$z = L(x,y) = -6 - 5(x-3) + 12(y+4)$

3. Set up **but do not solve** the following problems. As part of setting these problems up, you should list the unknowns and the equations that you would need to use to find them. You should also do all of the derivative calculations.

(a) Maximize $f(x,y) = xe^{x^3y^2} - y\sin(xy)$
Subject to $g(x,y) = x^6 + y^6 + 3xy - 4x^2y^3 = 1000$.

$\nabla f = \lambda \nabla g$

3 unknowns

(λ, x, y)

3 eqn!s

$$e^{x^3y^2} + 3x^3y^2e^{x^3y^2} - y^2\cos(xy) = \lambda(6x^5 + 3y - 8xy^3)$$

$$2x^4y\,e^{x^3y^2} - \sin(xy) - xy\cos(xy) = \lambda(6y^5 + 3x - 12x^2y^2)$$

$$x^6 + y^6 + 3xy - 4x^2y^3 = 1000$$

(b) Maximize $F(x,y,z) = x^2y^3z^4$
Subject to $G(x,y,z) = x^2 + y^2 + z^2 = 10^2$
and $H(x,y,z) = 3x + 4y + 5z = 0$.

5 unknowns

(λ, μ, x, y, z)

$\nabla F = \lambda \nabla G + \mu \nabla H$

5 eqn!s

$$2xy^3z^4 = \lambda 2x + \mu 3$$

$$3x^2y^2z^4 = \lambda 2y + \mu 4$$

$$4x^2y^3z^3 = \lambda 2z + \mu 5$$

$$x^2 + y^2 + z^2 = 10^2$$

$$3x + 4y + 5z = 0$$

4. For the curve $\vec{r}(t) = (\sin(3t),\ \cos(3t),\ 4t)$, find the unit tangent, the unit normal, the unit binormal, the curvature, and the tangential and the normal acceleration vectors.

$$\vec{r}\,'(t) = (3\cos(3t), -3\sin(3t), 4)$$

$$\|\vec{r}\,'(t)\| = 5$$

$$\vec{r}\,''(t) = (-9\sin(3t), -9\cos(3t), 0)$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = (36\cos(3t), -36\sin(3t), -27)$$

$$= 9(4\cos(3t), -4\sin(3t), -3)$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = 9 \cdot 5 = 45$$

$$\vec{B}(t) = \frac{(4\cos(3t), -4\sin(3t), -3)}{5}$$

$$a_T = 0, \quad a_N = 9, \quad \kappa = \frac{9}{25}$$

$$\vec{T}(t) = \left(\frac{3}{5}\cos(3t), -\frac{3}{5}\sin(3t), \frac{4}{5}\right)$$

$$\vec{N}(t) = (-\sin(3t), -\cos(3t), 0)$$

5. For the function $f(x, y) = 2x^2y + 16y^3 + x^2 + 20y^2$ find and classify all of the critical points.

$$\nabla f = (4xy + 2x,\ 2x^2 + 48y^2 + 40y) = (0,0)$$

$$f_{xx} = 4y + 2,\quad f_{xy} = 4x,\quad f_{yy} = 96y + 40$$

$$2x(2y+1) = 0 \Rightarrow x = 0 \text{ or } y = -\tfrac{1}{2}$$

$$x = 0 \Rightarrow (6y+5)y = 0 \Rightarrow y = 0 \text{ or } -\tfrac{5}{6}$$

$$y = -\tfrac{1}{2} \Rightarrow 2x^2 + 12 - 20 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$

CPs: $(0,0), (0, -\frac{5}{6}), (2, -\frac{1}{2}), (-2, -\frac{1}{2})$

$D(0,0) = 80 > 0$, $f_{xx}(0,0) > 0 \Rightarrow (0,0)$ is a loc. min.

$D(0, -\frac{5}{6}) = (-\frac{10}{3} + 2)(-80 + 40) > 0$, $f_{xx}(0, -\frac{5}{6}) < 0$
$\Rightarrow (0, -\frac{5}{6})$ is a loc. max

$D(2, -\frac{1}{2}) = D(-2, -\frac{1}{2}) = -16 \cdot 2^2 < 0$

Both $(2, -\frac{1}{2})$ & $(-2, -\frac{1}{2})$ are saddles

6. Find the maximum and the minimum of the function

$$f(x, y) = x^2 - 2x + y^2 + 2y$$

in the region given by

$$g(x, y) = x^2 + y^2 \leq 18.$$

Show your work carefully in this problem.

Part I: $g(x,y) < 18$. Set $\nabla f = 0$.

$\nabla f = (2(x-1), 2(y+1)) = (0,0) \Rightarrow x = 1, y = -1$

$f(1,-1) = -2$

Part II: $g(x,y) = 18$. Use LM's Set $\nabla f = \lambda \nabla g$

$$\left.\begin{aligned} 2(x-1) &= \lambda \cdot 2x \\ 2(y+1) &= \lambda \cdot 2y \\ x^2 + y^2 &= 18 \end{aligned}\right\} \text{3eqn's} \quad \text{3 unknowns: } (\lambda, x, y)$$

$$1 - \frac{1}{x} = \frac{x-1}{x} = \frac{y+1}{y} = 1 + \frac{1}{y} \Rightarrow y = -x$$

$2x^2 = 18 \Rightarrow x = \pm 3$ $(3,-3)$ & $(-3,3)$ need to be tested

$f(3,-3) = 6$, $f(-3,3) = 30$

Abs. Min: $(1,-1)$

Abs. Max: $(-3,3)$

7. Short answers ...

(a) If f is a function of x, y, and z, and x, y, and z, are each functions of q, r, s, and t, then what is

$$\frac{\partial f}{\partial t}?$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

(b) Which has larger curvature, a circle with radius 3 or a circle with radius 30?

The circle w/ $r=3$ has larger curvature

(c) What is the curvature of a circle with radius 7?

$$\frac{1}{7}$$

(d) Give the definition of what it means for a function to be continuous at a point.

$f(x,y)$ is cont. at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(e) For the set $\underbrace{x^2 + y^4 + z^6 + xyz}_{F(x,y,z)} = 2$ write down the tangent plane at the point $(1, -1, 1)$.

$$\nabla F = (2x+yz,\ 4y^3+xz,\ 6z^5+xy)$$

$$\nabla F(1,-1,1) = (1, 5, 5)$$

$$(1,5,5) \cdot (x-1,\ y+1,\ z-1) = 0$$