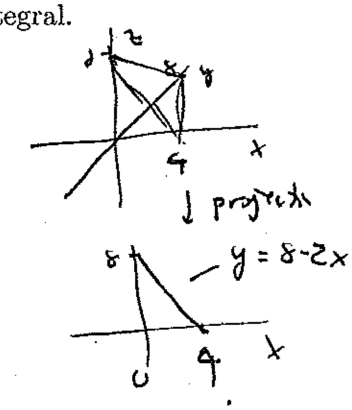
Short answer questions (8 points each):

1. Express the triple integral of the function $x^2 - y + z$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 8$. Do not evaluate the iterated integral.

$z = 8 - 2x - y$

$$\int_0^4 \int_0^{8-2x} \int_0^{8-2x-y} x^2 - y + z \, dz \, dy \, dx$$



2. Is the vector field $\vec{F}(x, y, z) = \langle xy, xz, yz \rangle$ conservative? If so, find a potential function. If not, demonstrating that it is not completely answers the question

Test by finding curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix} = (z-x)\vec{i} - (0-0)\vec{j} + (z-x)\vec{k}$$

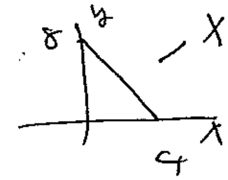
$$\neq \vec{0}$$

so it's <u>not</u> conservative

Other orders of integration for #1

$$\int_0^8 \int_0^{4-y/2} \int_0^{8-2x-y} x^2-y+z \, dz \, dx \, dy$$

proj to xy plane



$X = 4 - y/2$

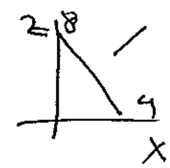
$$\int_0^4 \int_0^{8-2x} \int_0^{8-2x-z} x^2-y+z \, dy \, dz \, dx$$

$$\int_0^8 \int_0^{4-z/2} \int_0^{8-2x-z} x^2-y+z \, dy \, dx \, dz$$

back;

$y = 8-2x-z$

proj to xz plane



$z = 8-2x$

$x = 4 - z/2$

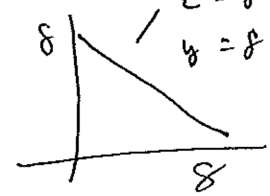
$$\int_0^8 \int_0^{8-y} \int_0^{4-y/2-z/2} x^2-y+z \, dx \, dz \, dy$$

$$\int_0^8 \int_0^{8-z} \int_0^{4-y/2-z/2} x^2-y+z \, dx \, dy \, dz$$

right side;

$X = 4 - y/2 - z/2$

proj to y-z plane



$z = 8-y$

$y = 8-z$

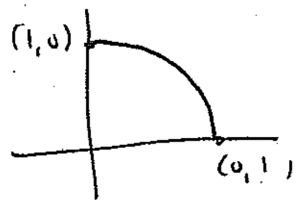
Short answer questions, continued.

3. Let C be the portion of the circle of radius 1 about the origin lying in the first quadrant beginning at $(0,1)$ and ending at $(1,0)$. Give a parametrization of C and use it to express the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

as an ordinary definite integral, where $\vec{F}$ is the vector field $\vec{F}(x,y) = \langle -1, x \rangle$. Do not evaluate the resulting definite integral.

parametrization by angle



$\vec{r}(t) = \langle \cos t, \sin t \rangle \quad t \in [0, \pi/2]$

$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

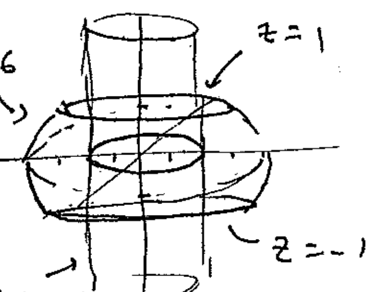
$\vec{F}(\vec{r}(t)) = \langle -1, \cos t \rangle$

$$\text{So } \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} (-1)(-\sin t) + (\cos t)(\cos t)\, dt$$

$$= \int_0^{\pi/2} \sin t + \cos^2 t \; dt$$

4. Express the volume of the region described in the next sentence as an iterated integral by using cylindrical coordinates. Do not evaluate the iterated integral.

The region lies between the planes $z = 1$ and $z = -1$, inside the sphere of radius 4 about the origin and outside the cylinder of radius 2 about the z-axis.



So $2 \le r \le \sqrt{16 - z^2}$

$-1 \le z \le 1$

$0 \le \theta \le 2\pi$

$dV = r\,dr\,dz\,d\theta$

$$V = \iiint_R dV = \int_0^{2\pi} \int_{-1}^{1} \int_2^{\sqrt{16-z^2}} r\,dr\,dz\,d\theta$$

#3 with parametrization by $t = y$

$$\vec{r}(t) = \langle \sqrt{1-t^2}, t \rangle \qquad t \in [0,1]$$

$$\vec{r}'(t) = \langle \tfrac{1}{2}(1-t^2)^{-1/2} \cdot (-2t), 1 \rangle$$

$$= \left\langle \frac{-t}{\sqrt{1-t^2}}, 1 \right\rangle$$

$$\vec{F}(\vec{r}(t)) = \langle -1, \sqrt{1-t^2} \rangle$$

So

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (-1)\frac{-t}{\sqrt{1-t^2}} + 1\sqrt{1-t^2}\, dt$$

$$= \int_0^1 \frac{t}{\sqrt{1-t^2}} + \sqrt{1-t^2}\, dt$$

Yet more short answer questions.

5. Find the divergence of $\vec{Q}(x,y,z) = ye^{z}\vec{i} + ze^{x}\vec{j} + xe^{y}\vec{k}$, and use it to decide whether or not the vector field could represent the velocity of a flow in an incompressible fluid.

$$\nabla \cdot \vec{Q} = \frac{\partial}{\partial x} ye^{z} + \frac{\partial}{\partial y} ze^{x} + \frac{\partial}{\partial z} xe^{y}$$

$$= 0 + 0 + 0 = 0$$

yes it could represent velocities in an incompressible fluid flow

6. Find the curl of the vector field $\vec{Q}(x,y,z) = ye^{z}\vec{i} + ze^{x}\vec{j} + xe^{y}\vec{k}$, and use it to decide whether the vector field is irrotational.

$$\nabla \times \vec{Q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{z} & ze^{x} & xe^{y} \end{vmatrix}$$

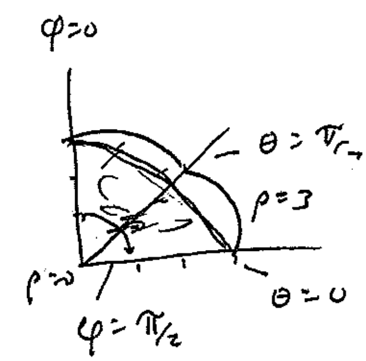
$$= (xe^{y} - e^{x})\vec{i} - (e^{y} - ye^{z})\vec{j} + (ze^{x} - e^{z})\vec{k}$$

No it is not irrotational.

Long questions (18 points each)

7. Let R be the region lying in the first octant (all coordinates positive) and inside the sphere of radius 3 about the origin. Use spherical coordinates to express the following triple integral as an iterated integral and evaluate the resulting iterated integral.

$$\iiint_R \frac{z}{x^2+y^2+z^2}\, dV$$



$x^2+y^2+z^2=\rho^2$

$z=\rho\cos\varphi$

$dV=\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \frac{\rho\cos\varphi\,\cancel{\rho^2}\sin\varphi}{\cancel{\rho^2}}\, d\rho\, d\varphi\, d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho\cos\varphi\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$=\int_0^{\pi/2} d\theta \int_0^{\pi/2}\cos\varphi\sin\varphi\, d\varphi \int_0^3 \rho\, d\rho$$

Since in ρ,φ,θ parameter space its a rect solid and the integrand factors as 1-vbl functions

let $u=\sin\varphi$
$du=\cos\varphi\, d\varphi$
$\varphi=0 \Rightarrow u=0 \quad \varphi=\pi/2 \Rightarrow u=1$

$$=\pi/2\int_0^1 u\, du \cdot \frac{\rho^2}{2}\Big|_0^3$$

$$=\frac{\pi}{2}\cdot\frac{u^2}{2}\Big|_0^1\cdot\frac{\rho^2}{2}\Big|_0^3=\frac{\pi}{2}\cdot 1/2\cdot 9/2=\frac{9\pi}{8}$$

8. Find the average value of the function $w(x, y) = x^2$ on the disk of radius 1 centered at the origin (in the xy-plane).

Area $= \pi \cdot 1^2 = \pi$

So Avg value $= \frac{1}{\pi} \iint_R x^2 \, dA$

polar coords seems good for this: since region is
$0 \le \theta \le 2\pi$
$0 \le r \le 1$

So $x^2 = r^2 \cos^2\theta$
$dA = r \, dr \, d\theta$

Avg $= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \, dr \, d\theta$

$= \frac{1}{\pi} \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^1 r^3 \, dr$

$= \frac{1}{\pi} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right] \Big|_0^{2\pi} \cdot \frac{r^4}{4} \Big|_0^1$ using #17 in the integral table

$= \frac{1}{\pi} \left[\frac{2\pi}{2} + 0 - (0 - 0) \right] \cdot \left(\frac{1}{4} - 0 \right)$

$= \frac{1}{\pi} \cdot \pi \cdot \frac{1}{4} = \frac{1}{4}$

#8 in rectangular coörds : region is

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$
$$-1 \le x \le 1$$

So

$$\text{Avg} = \frac{1}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 \, dy \, dx$$

$$= \frac{1}{\pi} \int_{-1}^{1} x^2 y \Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \frac{1}{\pi} \int_{-1}^{1} 2x^2 \sqrt{1-x^2} \, dx = \frac{2}{\pi} \int_{-1}^{1} x^2 \sqrt{1-x^2} \, dx$$

by form 9 on integral table

$$= \frac{2}{\pi} \left[-\frac{x}{4}(1-x^2)^{3/2} + \frac{1}{8}\left(x\sqrt{1-x^2} + \arcsin x \right) \right]_{x=-1}^{x=1}$$

doubled since its an odd function and we're subtracting its value at -1 from its value at 1

$$= \frac{4}{\pi} \left[-\frac{1}{4} \cdot 0 + \frac{1}{8} \cdot \left(1 \cdot 0 + \frac{\pi}{2} \right) \right]$$

evaluate at $x=1$

$$= \frac{4}{\pi} \cdot \frac{1}{8} \cdot \frac{\pi}{\cancel{2}} = \frac{1}{4}$$

9. Find the average value of the function $w(x, y) = x^2$ on the circle of radius 1 centered at the origin (in the xy-plane). (Just on the circle, not the whole disk – that was the previous question.)

length of curve = circumference $2\pi \cdot 1 = 2\pi$

So avg value $= \dfrac{1}{2\pi} \displaystyle\int_C x^2 \, ds$

parametrize by the angle

$\vec{r}(t) = \langle \cos t, \sin t \rangle \qquad t \in [0, 2\pi]$

so $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

so $ds = dt$

$x^2 = \cos^2 t$

$\therefore$ Avg value $= \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \cos^2 t \, dt$

$= \dfrac{1}{2\pi} \left[\dfrac{t}{2} + \dfrac{\sin 2t}{4} \right] \Bigg|_0^{2\pi}$ by fn 17a integ-table.

$= \dfrac{1}{2\pi} \left[\dfrac{2\pi}{2} + 0 - (0 + 0) \right]$

$= \dfrac{1}{2}$

10. A large magnetic sheet (idealized as) occupying the plane $z = 4$ (in a convenient coordinate system for the problem with distances in meters) exerts a force of $\frac{1}{(4-z)^2}\vec{k}$ Newtons on an iron particle located at (x, y, z) when $z < 4$. If the force causes the iron particle, constrained in a helical tube, to move from $(3, 0, 0)$ to $(3, 0, 2)$ along the curve given parametrically by $\vec{r}(t) = \langle 3\cos t, 3\sin t, t/\pi\rangle$. Find the work done by the force in Nm.

It will be easier if it's conservative. Let's test:

if $\vec{F} = \dfrac{1}{(4-z)^2}\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (4-z)^{-2} \end{vmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + (0-0)\vec{k}$$

So its conservative   Find a potential function:

$$f = \int (4-z)^{-2}\,dz = +(4-z)^{-1} + C(x,y)$$

plainly $f = +(4-z)^{-1}$ works since its x and y partials are 0

So $$W = \int_C \vec{F}\cdot d\vec{r} = f(3,0,2) - f(3,0,0)$$

$$= +(4-2)^{-1} - +(4-0)^{-1}$$

$$= +\frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{4} \text{ Nm}$$

#10 by finding the line integral directly

curve $\vec{r}(t) = \langle 3\cos t, 3\sin t, t/\pi \rangle \quad t \in [0, 2\pi]$

$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 1/\pi \rangle$

$F(\vec{r}(t)) = \dfrac{1}{(4 - t/\pi)^2}\vec{k}$

So

$$W = \int_0^{2\pi} \langle 0, 0, (4 - t/\pi)^{-2} \rangle \cdot \langle -3\sin t, 3\sin t, 1/\pi \rangle \, dt$$

$$= \int_0^{2\pi} \frac{1}{\pi}(4 - t/\pi)^{-2} \, dt$$

let $u = 4 - t/\pi$

$du = -1/\pi \, dt$

$\therefore -du = 1/\pi \, dt$

$$= \int_4^2 -u^{-2} \, du$$

$t = 0 \Rightarrow u = 4$

$t = 2\pi \Rightarrow u = 4 - \frac{2\pi}{\pi} = 2$

$$= \int_2^4 u^{-2} \, du = -u^{-1} \Big|_2^4$$

$$= -\frac{1}{4} - (-1/2) = 1/4 \text{ Nm}$$