Short answer questions (12 points each):

1. Recall that Green's Theorem can be used to express the area of a region as a line integral over the boundary of the region (Stewart gave three formulas, but any vector field for which the integrand in the double integral of Green's theorem equals the constant function 1 will work). Express the area inside one leaf of the four leafed rose given in polar coordinates by $r = \cos 2\theta$ as a line integral, and write the line integral as an ordinary integral by using the parametrization provided. Do not evaluate the resulting integral. The boundary of one leaf is given parametrically by

$$\vec{r}(t) = \cos(2t)\cos(t)\vec{\imath} + \cos(2t)\sin(t)\vec{\jmath} \quad t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$A = \oint_{\partial R} x\,dy$

using the parametrization

$x = \cos(2t)\cos(t)$

$dy = -2\sin(2t)\sin(t) + \cos(2t)\cos(t)\,dt$

So

$$A = \int_{\pi/4}^{3\pi/4} \cos(2t)\cos(t)\left[-2\sin(2t)\sin(t) + \cos(2t)\cos(t)\right] dt$$

2. The vector field on the plane $\vec{H}(x,y) = (3x^2 - 3y)\vec{\imath} + (12y - 3x)\vec{\jmath}$ is conservative. Find a potential function h such that $\nabla h = \vec{H}$.

h satisfies the following equations given by partial antidifferentiation:

$h = \int 3x^2 - 3y\,dx = x^3 - 3xy + C(y)$

$h = \int 12y - 3x\,dy = 6y^2 - 3xy + K(x)$

$\therefore$ letting $C(y) = 6y^2$, $K(x) = x^3$ we find

$h(x,y) = x^3 - 3xy + 6y^2$

is a potential function for $\vec{H}$

More short answer questions:

3. Express the flux of the vector field $\vec{H}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ through the portion of the graph of $z = x^2 - y^2$ lying inside the cylinder $x^2 + y^2 = 16$ as an ordinary iterated integral. Do not evaluate the iterated integral.

parametrize the surface as

$\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle \qquad (u,v) \in \{(u,v) \mid u^2+v^2 \leq 16\}$

so $\vec{r}_u(u,v) = \langle 1, 0, 2u \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, -2v \rangle$

$\vec{r}_u \times \vec{r}_v (u,v) = \langle 2u, 2v, 1 \rangle$

$\vec{H}(\vec{r}(u,v)) = \langle u^2, v^2, (u^2-v^2)^2 \rangle$

$$\iint_\Sigma \vec{H} \cdot d\vec{S} = \int_{-4}^{4} \int_{-\sqrt{16-v^2}}^{\sqrt{16-v^2}} 2u^3 + 2v^3 + (u^2-v^2)^2 \, du\, dv$$

4. Express the average value of the function $f(x,y,z) = x^2 + y^2 + z^2$ on the portion of the graph of $z = x^2 - y^2$ lying inside the cylinder $x^2 + y^2 = 16$ as the ratio of two ordinary iterated integrals. Do not evaluate (either of) the integrals.

Same parametrization as above:

but $dS = |\langle 2u, 2v, 1 \rangle| \, dA = \sqrt{4u^2+4v^2+1}\, du\, dv$

$f(\vec{r}(u,v)) = u^2 + v^2 + (u^2-v^2)^2$

$$\text{Avg} = \frac{\displaystyle\int_{-4}^{4} \int_{-\sqrt{16-v^2}}^{\sqrt{16-v^2}} [u^2+v^2+(u^2-v^2)^2]\sqrt{4u^2+4v^2+1}\, du\, dv}{\displaystyle\int_{-4}^{4} \int_{-\sqrt{16-v^2}}^{\sqrt{16-v^2}} \sqrt{4u^2+4v^2+1}\, du\, dv}$$

Yet more short answer questions:

5. Find the flux of the vector field $\vec{F}(x,y,z) = \vec{\imath} + 3\vec{\jmath}$ through the sphere of radius 2 about the origin.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} 1 + \frac{\partial}{\partial y} 3 + \frac{\partial}{\partial z} 0 = 0$$

$$\therefore \iint_{\Sigma} \vec{F} \cdot d\vec{s} = \iiint_{B} 0 \, dV = 0$$

(B the ball of radius 2 about origin).

6. Give a vector equation for the line passing through the point $(1, -3, \frac{1}{4})$ and perpendicular to the plane $3x - \frac{1}{3}y + z = \frac{4}{5}$.

direct vector is normal to plane so $\langle 3, -\frac{1}{3}, 1 \rangle$ works

line is

$$\vec{r}(t) = t\langle 3, -\tfrac{1}{3}, 1 \rangle + \langle 1, -3, \tfrac{1}{4} \rangle$$

$$= \langle 3t+1, -\tfrac{1}{3}t - 3, t + \tfrac{1}{4} \rangle$$

7. Write, but do not solve, the system of (scalar) equations in the variables x, y, z, and λ that you would need to solve in finding the maximum and minimum values of $f(x,y,z) = xy^2z^3$ subject to the constraint $x^2 + 4y^2 + 9z^2 = 36$ by the method of Lagrange multipliers.

let $g(x,y,z) = x^2 + 4y^2 + 9z^2$

then at extrema $\nabla f = \lambda \nabla g$

$$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$$

$$\nabla g = \langle 2x, 8y, 18z \rangle$$

So one would need to solve

$$y^2z^3 = 2\lambda x$$

$$2xyz^3 = 8\lambda y$$

$$3xy^2z^2 = 18\lambda z$$

$$x^2 + 4y^2 + 9z^2 = 36$$

Matching (20 points)

8. Ten formulas are given in the left column. The ten lettered items in the right column are descriptions or other formulas, each of which applies to exactly one item in the left column in the sense of correctly describing it or being equal to it. Put the letter of the item in the right column which applies to each item in the left column in the blank provided next to the corresponding item in the left column. Please uses block capital letters.

H ______ $\iint\limits_{\partial R} \langle x, y, z \rangle \bullet d\vec{S}$

C ______ $\int\limits_{C} \vec{F} \bullet d\vec{r}$

I ______ $t^3\vec{i} + \cos(t)\vec{j} + \sin(t)\vec{k}$

G ______ $x^3\vec{i} + \cos(z)\vec{j} + \sin(y)\vec{k}$

J ______ $\frac{\vec{r}_u(u,v)\times\vec{r}_v(u,v)}{|\vec{r}_u(u,v)\times\vec{r}_v(u,v)|}$

E ______ $\frac{\nabla r}{|\nabla r|}$

D ______ $x\vec{i} + y\vec{j} - \vec{k}$

A ______ $3\vec{i} + \vec{j} - \vec{k}$

F ______ $\iint\limits_{R} y - x \, dA$

B ______ $\int\limits_{C} f ds$

A. an irrotational vector field, which could also represent the flow of an incompressible fluid

B. useful in finding the average value of a function on a curve

C. useful in finding the work done by a force varying from place to place moving a particle along a curve

D. an irrotational vector field which could not represent the flow of an incompressible fluid

E. gives unit normal vectors to level surfaces

F. $\int\limits_{\partial R} xydx + xydy$

G. a vector field which is neither irrotational nor represents the flow of an incompressible fluid

H. $\iiint\limits_{R} 3dV$

I. a curve lying on the cylinder $y^2 + z^2 = 1$

J. gives unit normal vectors to a parametric surface

Long questions (24 points each)

9. Find the length of the curve given parametrically by

$$\vec{r}(t) = \left\langle \frac{t^3}{3}, \frac{t^2}{\sqrt{2}}, t \right\rangle \quad t \in [0, 5]$$

(Hint: if you think you should have been given an integral table for this one, you've done something wrong. Find it and fix it.)

$$\vec{r}'(t) = \left\langle t^2, \frac{2t}{\sqrt{2}}, 1 \right\rangle = \langle t^2, \sqrt{2}t, 1 \rangle$$

$$|\vec{r}(t)| = \sqrt{t^4 + 2t^2 + 1} = t^2 + 1$$

so

$$ds = t^2 + 1 \, dt$$

$$l = \int_0^5 \cancel{t}^2 + 1 \, dt = \frac{t^3}{3} + t \Big|_0^5$$

$$= \frac{125}{3} + 5 - (0 + 0)$$

$$= \frac{140}{3}$$

10. Find the critical points of the function $f(x,y) = x^3 - 3xy + 6y^2$ and use the second deriviative test to determine whether each is a local maximum, a local minimum or a saddle point.

at c.p.

$f_x(x,y) = 3x^2 - 3y = 0$

$f_y(x,y) = -3x + 12y = 0$

$\therefore\ 12y = 3x \quad 4y = x$

$3(4y)^2 - 3y = 0$

$3y(16y - 1) = 0$

$y = 0$ or $y = \frac{1}{16}$

so c.p.'s: $(0,0)$, $\left(\frac{1}{4}, \frac{1}{16}\right)$

$f_{xx}(x,y) = 6x$

$f_{xy}(x,y) = f_{yx}(x,y) = -3$

$f_{yy}(x,y) = 12$

$$D(x,y) = \begin{vmatrix} 6x & -3 \\ -3 & 12 \end{vmatrix} = 72x - 9$$

$D(0,0) = -9 < 0$ $\quad\therefore (0,0)$ is a saddle pt

$D\left(\frac{1}{4}, \frac{1}{16}\right) = 72 \cdot \frac{1}{4} - 9$

$= 18 - 9 = 9 > 0$ ← all concavities same direction

$f_{yy}\left(\frac{1}{4}, \frac{1}{16}\right) = 12 > 0$ ← concave up in y direction

$\therefore \left(\frac{1}{4}, \frac{1}{16}\right)$, is a local minimum

11. Recall that the helicoid is the surface given parametrically by

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$$

(a) Find the equation of the plane tangent to the helicoid at $(0,2,\frac{\pi}{2})$. $= \vec{r}(2, \pi/2)$

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u\rangle$

So at $(0, 2, \pi/2)$ $\quad \langle \sin \pi/2, -\cos \pi/2, 2\rangle = \langle 1, 0, 2\rangle$ is

normal to the tan pl

So $\langle 1, 0, 2\rangle \cdot \langle x, y-2, z-\pi/2\rangle = 0$

$x + 2z - \pi = 0 \qquad x + 2z = \pi$ is the tan plane

(b) Find the flux of the vector field $x\vec{i} + y\vec{j}$ through the portion of the helicoid with $0 \le u \le 2$ and $0 \le v \le 4\pi$ (in the direction given by the normals determined by the parametrization as given).

So we have $\vec{r}_u \times \vec{r}_v$ in part (a)

If $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + 0\vec{k}$

$\vec{F}(\vec{r}(u,v)) = u\cos v\,\vec{i} + u\sin v\,\vec{j}$

$$\iint_\Sigma \vec{F}\cdot d\vec{S} = \int_0^{4\pi}\int_0^2 u\cos v\sin v + u\sin v(-\cos v)\,du\,dv$$

$$= \int_0^{4\pi}\int_0^2 0\,du\,dv = 0$$

12. Write iterated integrals equal to the triple integral

$$\iiint_R z\,dV$$

where R is the ball of radius 3 about the origin, using each of rectangular, cylindrical and spherical coordinates. Briefly explain which you would prefer to evaluate and why. You do not need to evaluate any of the interated integrals.

rect $$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} z\,dz\,dy\,dx$$

Cylindrical: $dV = r\,dr\,dz\,d\theta$ (or with diff't ordering)

$z \to z$. bounds are $-\sqrt{9-r^2} \le z \le \sqrt{9-r^2}$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

So $$\int_0^{2\pi}\int_0^{3}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} zr\,dz\,dr\,d\theta$$

Spherical $dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$ bounds $0 \le \rho \le 3$

$z = \rho\cos\varphi$ $0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho^3 \cos\varphi \sin\varphi\,d\rho\,d\varphi\,d\theta$$

Sph: easier than others, int [illegible] [illegible] [illegible].