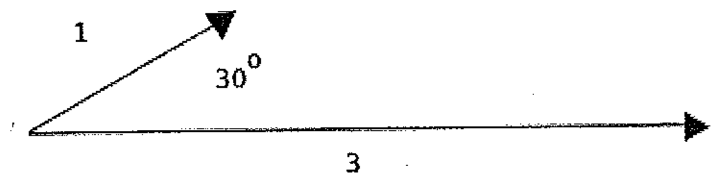
Short answer questions (6 points each):

Let the diagonal vector in the picture be denoted **u** and the horizontal vector be denoted **v**, with the magnitudes and angle between the vectors as indicated.



1. Find $|\mathbf{u} \times \mathbf{v}|$

$$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin\theta$$
$$= 1 \cdot 3 \cdot \tfrac{1}{2} = \tfrac{3}{2}$$

2. Regarding the page as a plane in 3-space, does $\mathbf{u} \times \mathbf{v}$ point up out of the page toward you, or down toward the desk or table?

down

3. Find $\mathbf{u} \cdot \mathbf{v}$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$
$$= 1 \cdot 3 \cdot \tfrac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

4. Suppose **v** is taken to be a displacement vector with magnitude in meters and **u** is taken to be a force with magnitude in Newtons. Write a brief word problem for which solving question 3 gives the computational work in solving the word problem, and tell the physical units that should be applied to the answer in 3 to make it a solution to the word problem.

A force of 1 Newton acting at 30° above horizontal drags a box 3 meters. Find the work done. The units are N·m.

Short answer questions, continued.

5. Find the equation of the plane perpendicular to the line given parametrically by $\mathbf{r}(t) = t\langle 3, \frac{2}{5}, -1\rangle + \langle 2, 5, \frac{1}{3}\rangle$ and passing through the point $\langle 2, -1, 3\rangle$.

So the direction vector of the line $\langle 3, 2/5, -1\rangle$ must be normal to the plane

$$\langle 3, 2/5, -1\rangle \cdot \left(\langle x, y, z\rangle - \langle 2, -1, 3\rangle\right) = 0$$

$$\langle 3, 2/5, -1\rangle \cdot \langle x-2, y+1, z-3\rangle = 0$$

$$3x - 6 + 2/5\,y + 2/5 - z + 3 = 0$$

$$3x + 2/5\,y - z = 13/5$$

6. Find $\langle 3, -\frac{1}{2}, 0\rangle \times \langle \frac{3}{2}, 2, -1\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1/2 & 0 \\ 3/2 & 2 & -1 \end{vmatrix} = (1/2 - 0)\,\vec{i} - (-3 - 0)\,\vec{j} + (6 + 3/4)\,\vec{k}$$

$$= \frac{1}{2}\vec{i} + 3\vec{j} + \frac{27}{4}\vec{k}$$

Yet more short anwer questions:

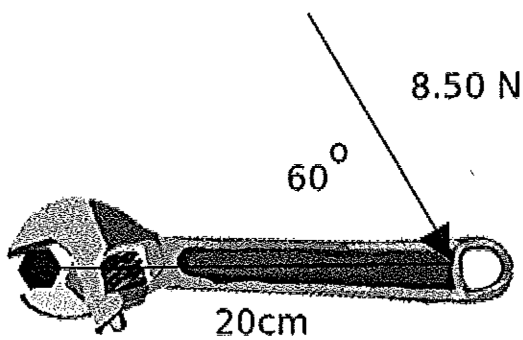
7. Find the angle between the planes $5x + 3y - 2z = 10$ and $2x - y + z = 0$. As you are not allowed calculators, the correct answer will necessarily be in the form of some inverse trig function applied to a number.

$$\Theta = \arccos \frac{\langle 5,3,-2\rangle \cdot \langle 2,-1,1\rangle}{|\langle 5,3,-2\rangle||\langle 2,-1,1\rangle|}$$

$$= \arccos \frac{10-3-2}{\sqrt{25+9+4}\sqrt{4+1+1}} = \arccos \frac{5}{\sqrt{38}\sqrt{6}}$$

$$= \arccos \frac{5}{\sqrt{228}} = \arccos \frac{5}{2\sqrt{57}}$$

8. What is the magnitude of the torque applied to the bolt when the force is applied to the wrench as illustrated?



$$\vec{\tau} = \vec{r} \times \vec{F}$$

so $$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\Theta$$

$$= (.20\ m)(8.50\ N)\ \sqrt{3}/2 = \frac{1.7\sqrt{3}}{2}\ N\cdot m$$

Long questions, point values follow the question number in parentheses.

9. (16)

(a) Find an equation for the plane passing through the points $(1, 0, 1)$, $(1, 2, 3)$ and $(-1, 2, 3)$.

Let $\vec{p} = \langle 1, 0, 1 \rangle$, $\vec{q} = \langle 1, 2, 3 \rangle$, $\vec{r} = \langle -1, 2, 3 \rangle$

so $\vec{q} - \vec{p} = \langle 0, 2, 2 \rangle$

and $\vec{r} - \vec{p} = \langle -2, 2, 2 \rangle$ are displacements $\parallel$ to the plane

Their cross product will be a normal vector:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 2 \\ -2 & 2 & 2 \end{vmatrix} = (4-4)\vec{i} - (0-(-4))\vec{j} + (0-(-4))\vec{k}$$
$$= -4\vec{j} + 4\vec{k}$$

so $\langle 0, -4, 4 \rangle \cdot (\langle x, y, z \rangle - \underbrace{\langle 1, 0, 1 \rangle}_{\text{point on plane}}) = 0$ is an equation for the plane.

(b) Find the distance from the point $(4, 0, -1)$ to the plane in part (a).

$\vec{x} = \langle 4, 0, -1 \rangle$

$\vec{n} = \langle 0, -4, 4 \rangle$

$\vec{p} = \langle 1, 0, 1 \rangle$

$\vec{x} - \vec{p} = \langle 3, 0, -2 \rangle$

$$\text{distance} = \operatorname{comp}_{\vec{n}} (\vec{x} - \vec{p})$$

$$= \frac{|\langle 0, -4, 4 \rangle \cdot \langle 3, 0, -2 \rangle|}{|\langle 0, -4, 4 \rangle|}$$

$$= \frac{|0 + 0 - 8|}{\sqrt{16 + 16}} = \frac{8}{4\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

or use formula Stewart gave

10. (16) Find

$$\int_0^3 3t\mathrm{i} + t^2\mathrm{j} - \frac{1}{t^2+16}\mathrm{k}\,dt$$

and give a physical interpretation if the integrand represents velocity in m/sec and the variable of integration represents time elapsed in seconds.

Let's integrate the component functions separately:

$$\int_0^3 3t\,dt = \frac{3t^2}{2}\Big|_0^3 = \frac{27}{2} \qquad \int_0^3 t^2\,dt = \frac{t^3}{3}\Big|_0^3 = 9$$

$$\int_0^3 \frac{1}{t^2+16}\,dt = \int_0^{?} \frac{4\sec^2\theta\,d\theta}{16\tan^2\theta+16}$$

let $t = 4\tan\theta$, $dt = 4\sec^2\theta\,d\theta$

$t=0 \Leftrightarrow \theta = 0$

$t = 3 \Leftrightarrow \theta = \arctan(3/4)$

$$= \frac{1}{4}\int_0^{\arctan(3/4)} \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{4}\arctan\left(\frac{3}{4}\right)$$

$\Uparrow$

<u>or</u> use $u = t/4$ after factoring out $\frac{1}{16}$

So

$$\int_0^3 3t\vec{i} + t^2\vec{j} - \frac{1}{t^2+16}\vec{k}\,dt$$

$$= \left\langle \frac{27}{2},\ 9,\ -\frac{1}{4}\arctan(3/4)\right\rangle$$

The value of the integral represents the displacement from the position of the particle at time 0 sec. to the position at time 3 sec.
(with units of meters)

11. (20) Consider the vector-valued function $\mathbf{r}(t) = 2\cos(\pi t)\mathbf{i} + 3\sin(\pi t)\mathbf{j}$ as a trajectory (i.e. giving the position of a particle at time t, say in seconds after some chosen time $t = 0$ as a displacement from the origin in some units, say cm).

(a) Find the velocity function $\mathbf{v}(t)$.

$$\vec{v}(t) = \vec{r}\,'(t) = -2\pi \sin(\pi t)\vec{i} + 3\pi \cos(\pi t)\vec{j}$$

(b) Find the acceleration function $\mathbf{a}(t)$.

$$\vec{a}(t) = \vec{r}\,''(t) = -2\pi^2 \cos(\pi t)\vec{i} - 3\pi^2 \sin(\pi t)\vec{j}$$

(c) Find the $\mathbf{v}(1)$ and $\mathbf{a}(1)$, the velocity and acceleration at time $t = 1$.

$$\vec{v}(1) = -2\pi \sin(\pi \cdot 1)\vec{i} + 3\pi \cos(\pi \cdot 1)\vec{j} = -3\pi\vec{j}$$

$$\vec{a}(1) = -2\pi^2 \cos(\pi)\vec{i} - 3\pi^2 \sin(\pi)\vec{j} = 2\pi^2\vec{i}$$

(d) Find $proj_{\mathbf{v}(1)}\mathbf{a}(1)$. (As you know from the lectures on topics that aren't on this exam, this is the part of the acceleration which is changing the speed at time $t = 1$.)

$$proj_{\vec{v}(1)}\vec{a}(1) = 0$$

since plainly $\vec{v}(1) \perp \vec{a}(1)$

(e) Find $orth_{\mathbf{v}(1)}\mathbf{a}(1)$. (As you know from the lectures on topics that aren't on this exam, this is the part of the acceleration which is changing the direction of travel at time $t = 1$.)

$$orth_{\vec{v}(1)}\vec{a}(1) = \vec{a}(1) = 2\pi^2\vec{j}$$

since plainly $\vec{v}(1) \perp \vec{a}(1)$