Short answer questions.

1. (6) Find both first partial derivatives of $f(x, y) = y\cos(x^2 + y)$

$f_x(x,y) = -y\sin(x^2+y)\cdot 2x = -2xy\sin(x^2+y)$

$f_y(x,y) = 1\cdot\cos(x^2+y) + y(-\sin(x^2+1)\cdot 1)$

$= \cos(x^2+y) - y\sin(x^2+y)$

2. (9) Find the gradient of $g(x, y, z) = x - yz^2$ at $(2, 1, -3)$

$\nabla g(x,y,z) = \langle 1, -z^2, -2yz\rangle$

$\nabla g(2,-1,3) = \langle 1, -3^2\ -2(-1)\cdot 3\rangle$

$= \langle 1, -9, 6\rangle$

3. (9) You are driving west at a constant speed when you crest a hill. At the moment you crest the hill, in what directions do the unit tangent vector, principal unit normal vector and binormal vector to your car's trajectory point? (Answer with "up", "down", compass directions, or phrases like "north and down toward the ground at a 45° angle".)

Unit tangent vector : west (and horizontal)

principal unit normal : down

binormal vector : South

Short answer questions, continued.

4. (8) Find the indicated limit, or explain why it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + 2y^2}$$

It does not exist.

Consider the 1 var limits approaching (0,0) along $y = mx$

$$\lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + 2m^2x^2} = \lim_{x\to 0} \frac{1-m^2}{1+2m^2} = \frac{1-m^2}{1+2m^2}$$

different one variable limits along different lines,

$\Rightarrow$ no 2 var limit.

5. (8) Find the indicated limit, or explain why it does not exist:

$$\lim_{(x,y)\to(1,0)} \frac{x^2 - y^2}{x^2 + 2y^2} = \frac{1-0^2}{1+2\cdot 0^2} = 1$$

Since $\frac{x^2-y^2}{x^2+2y^2}$ is continuous at (1, 0)

as all arithmetic ops are continuous on their domains

Yet more short answer questions.

6. (7) Set up, but do not evaluate, a definite integral whose value is the arc length of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

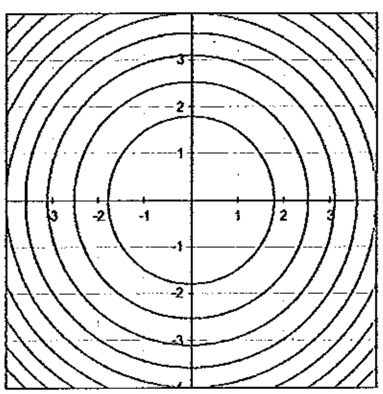
Hint: the ellipse is given parametrically by $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j}$ for $t \in [0, 2\pi)$.

$$s = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (4\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 16\cos^2 t}\, dt$$

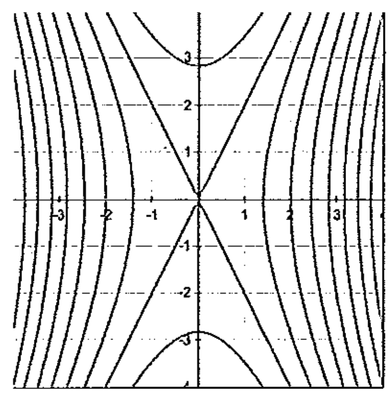
$\vec{r}'(t) = -3\sin t\,\vec{i} + 4\cos t\,\vec{j}$

7. (6) In each of the contour plots below, the origin is a critical point of the function. One is a saddle point, one is a local maximum. Which is which, and how do you know?



local max

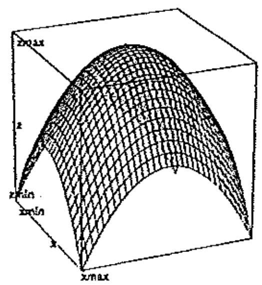
local max is (or can be) isolated in its level set.



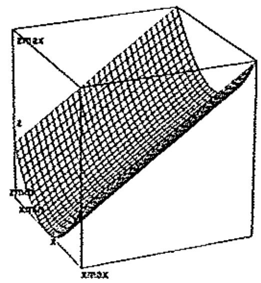
Saddle

Saddle [illegible], has a level set through it with more than one [illegible]

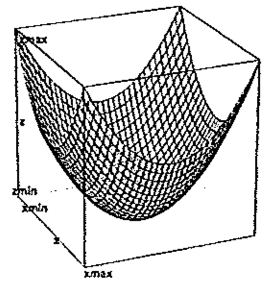
8. (12) Match the following graphs to the formula of the function of which they are the graph by putting the Roman numeral of the graph in the space provided next to the equation.



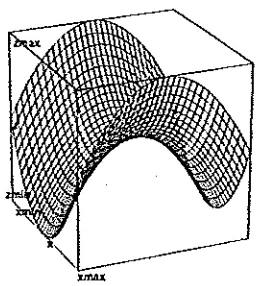
I.



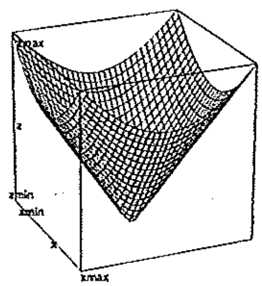
II.



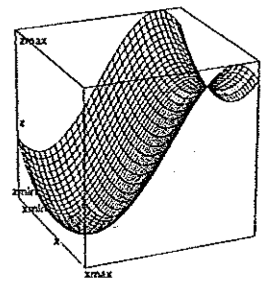
III.



IV.



V.



VI.

III $a(x, y) = x^2 + y^2$

IV $b(x, y) = x^2 - y^2$

II $c(x, y) = x^2 + y$

V $d(x, y) = sqrt(x^2 + y^2)$ $\sqrt{x^2+y^2}$

two typos

I $d(x, y) = 1 - x^2 - y^2$

VI $f(x, y) = x^2 - 2y^3 + 3y$

9. (18) Find and classify the critical points of

$$H(x, y) = x^3 - 12xy + 2y^2$$

$H_x(x,y) = 3x^2 - 12y$

$H_y(x,y) = -12x + 4y$

$H_{xx}(x,y) = 6x$

$H_{xy}(x,y) = H_{yx}(x,y) = -12$

$H_{yy}(x,y) = 4$

$$D = \begin{vmatrix} 6x & -12 \\ -12 & 4 \end{vmatrix} = 24x - 144$$

at c.p.

$3x^2 = 12y$

$12x = 4y$

$3x = y$

$3x^2 - 36x = 0$

$3x(x-12) = 0$

$x = 0$ or $x = 12$

$(0,0)$, $(12,36)$

$D(0,0) = 0 - 144 < 0$ $\therefore$ $(0,0)$ is a saddle pt

$D(12,36) = 24 \cdot 12 - 144 = 144 > 0$

and $H_{yy}(12,36) = 4 > 0$ $\therefore$ $(12,36)$ is a local min

10. (18) Use the method of Lagrange multipliers to find the maximum and minimum values of $P(x, y, z) = 4x - 3y + 12z$ on $\{(x, y, z) | x^2 + y^2 + z^2 = 169\}$, and the points at which they occur.

let $g(x,y,z) = x^2+y^2+z^2$

$\nabla P = \langle 4, -3, 12 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$\therefore$ at extrema $\langle 4, -3, 12 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$4 = 2\lambda x$
$-3 = 2\lambda y$ and $x^2+y^2+z^2 = 169$
$12 = 2\lambda z$

Solve for $\frac{1}{2\lambda}$:

$\frac{1}{2\lambda} = \frac{x}{4} = -\frac{y}{3} = \frac{z}{12}$

$\therefore$ $x = z/3$
$y = -z/4$

and $\therefore$ $(z/3)^2 + (-z/4)^2 + z^2 = 169$

$\frac{z^2}{9} + \frac{z^2}{16} + z^2 = 169$

$\frac{169z^2}{144} = \frac{16z^2 + 9z^2 + 144z^2}{144} = 169$

$\frac{z^2}{144} = 1$ $z^2 = 144$ $z = \pm 12$

So extrema occur at

$(4, -3, 12)$

or $(-4, 3, -12)$

$P(4,-3,12) = 16+9+144 = 169$ max

$P(-4,3,-12) = -16-9-144 = -169$ min.

11. (24) Consider the vector-valued function $\mathbf{r}(t) = 2\cos(t^2)\mathbf{i} + 2\sin(t^2)\mathbf{j}$ as a trajectory (i.e. giving the position of a particle at time t, say in seconds after some chosen time $t = 0$ as a displacement from the origin in some units, say cm).

(a) Find the velocity, speed, and acceleration functions.

$$\vec{v}(t) = \vec{r}\,'(t) = -4t\sin(t^2)\vec{i} + 4t\cos(t^2)\vec{j}$$

$$v(t) = \sqrt{16t^2\sin^2(t^2) + 16t^2\cos^2(t^2)} = 4t$$

$$\vec{a}(t) = \vec{r}\,''(t) = \left[-4\sin(t^2) - 8t^2\cos(t^2)\right]\vec{i} + \left[4\cos(t^2) - 8t^2\sin(t^2)\right]\vec{j}$$

(b) Find the function giving the unit tangent vector to the trajectory at each point in time and its derivative.

$$\vec{T}(t) = \frac{\vec{V}(t)}{v(t)} = -\sin(t^2)\vec{i} + \cos(t^2)\vec{j}$$

$$\vec{T}\,'(t) = -2t\cos(t^2)\vec{i} - 2t\sin(t^2)\vec{j}$$

(c) Find the function giving the curvature of the trajectory at each point in time.

$$k = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{\sqrt{4t^2\cos^2(t) + 4t^2\sin^2(t)}}{4t} = \frac{2t}{4t} = \frac{1}{2}$$

or notice the trajectory is a circle of radius 2 so $k = \frac{1}{2}$

(d) Express the acceleration as the sum of two vectors, one parallel to the velocity (whose magnitude gives the change in speed), the other orthogonal to the velocity.

$$\vec{a}(t) = [v(t)\cdot\vec{T}(t)]'$$

$$= v'(t)\cdot\vec{T}(t) + v(t)\vec{T}\,'(t)$$

$$= \underbrace{4(-\sin(t^2)\vec{i} + \cos(t^2)\vec{j})}_{\parallel \text{ to velocity}} + 4t(-2t\cos(t^2)\vec{i} - 2t\sin(t^2)\vec{j})$$

$$= \underbrace{(-4\sin(t^2)\vec{i} + 4\cos(t^2)\vec{j})}_{\parallel \text{ to velocity}} + \underbrace{(-8t^2\cos(t^2)\vec{i} - 8t^2\sin(t^2)\vec{j})}_{\perp \text{ to velocity}}$$