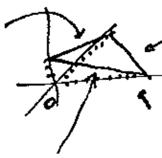
Short answer questions (7 points each):

1. Express the triple integral of the function x^2+y+z over the region bounded by the planes $x=0$, $y=0$, $z=0$, and $x+y+3z=9$ as an iterated integral. Do not evaluate the iterated integral – doing so will waste time needed to complete other questions.

$3z = 9-x-y$

$z = 3-\frac{x}{3}-\frac{y}{3}$

$y = 9-x-3z$



$x+y=9$

$y=9-x$

$z = 3-\frac{x}{3}$

$$\int_0^9 \int_0^{9-x} \int_0^{3-\frac{x}{3}-\frac{y}{3}} x^2+y+z \, dz\, dy\, dx$$

or
$$\int_0^9 \int_0^{3-\frac{x}{3}} \int_0^{9-x-3z} x^2+y+z^2 \, dy\, dz\, dx$$

or ... (all six orders of integration are feasible)

2. Is the vector field $\vec{F}(x,y,z) = \langle y, z, x\rangle$ conservative?

Test by finding the curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (0-1)\vec{i} - (1-0)\vec{j} + (0-1)\vec{k}$$

$$= -\vec{i} - \vec{j} - \vec{k} \neq \vec{0}$$

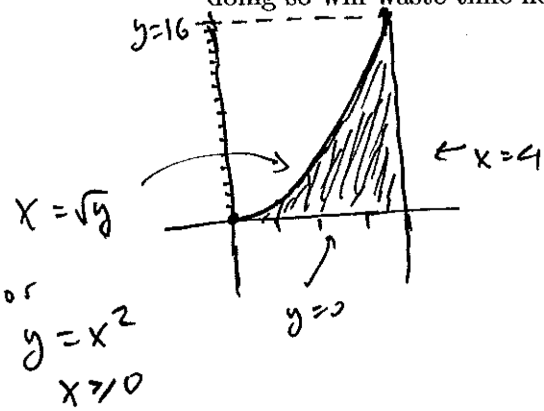
so no, it is not conservative.

Short answer questions, continued.

3. Consider the iterated integral

$$\int_0^{16} \int_{\sqrt{y}}^4 y \, dx dy$$

Draw the region over which this iterated integral represents the double integral of the function y and use your drawing to express the same double integral as an iterated integral with the order of integration reversed. You do not need to evaluate either iterated integral – doing so will waste time needed to complete other questions.



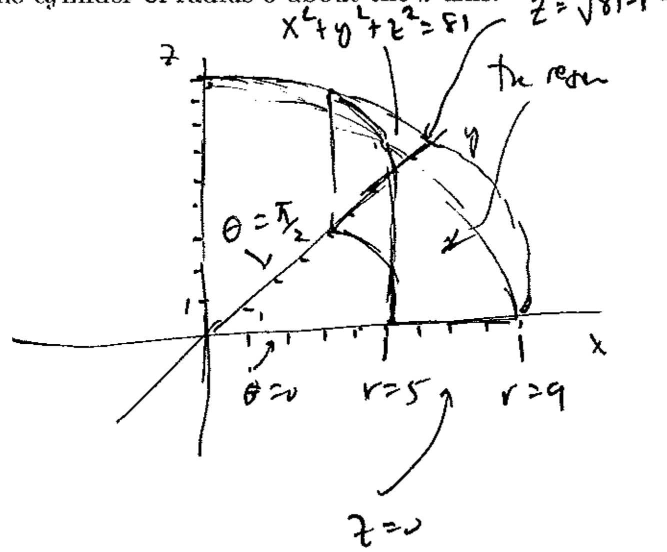
$$\int_0^4 \int_0^{x^2} y \, dy \, dx$$

4. Express the volume of the region described in the next sentence as an iterated integral by using cylindrical coordinates. Do not evaluate the iterated integral – doing so will, you guessed it, waste time needed to complete other questions.

The region lies in the first octant (all rectangular coordinates non-negative) inside the sphere of radius 9 about the origin and outside the cylinder of radius 5 about the z-axis.

$r^2+z^2=81$
$z=\sqrt{81-r^2}$

$$\int_0^{\pi/2} \int_5^9 \int_0^{\sqrt{81-r^2}} r \, dz \, dr \, d\theta$$



(or with the θ integral permuted to any other position)

Yet more short answer questions.

5. Find the divergence of $\vec{Q}(x,y,z) = e^x\vec{i} + e^y\vec{j} + e^z\vec{k}$, and use it to decide whether or not the vector field could represent the velocity of a flow in an incompressible fluid.

$$\nabla \cdot \vec{Q} = \frac{\partial}{\partial x} e^x + \frac{\partial}{\partial y} e^y + \frac{\partial}{\partial z} e^z$$

$$= e^x + e^y + e^z \neq 0$$

So no, it can't represent the velocity of an incompressible fluid flow.

6. Find the curl of the vector field $\vec{Q}(x,y,z) = e^x\vec{i} + e^y\vec{j} + e^z\vec{k}$, and use it to decide whether the vector field is irrotational.

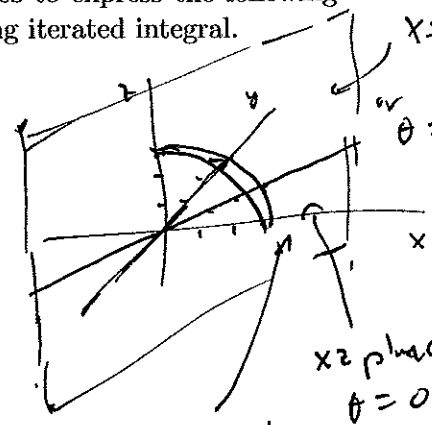
$$\nabla \times \vec{Q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & e^y & e^z \end{vmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + (0-0)\vec{k}$$

Yes. It is irrotational

Long questions (20 points each)

7. Let R be the region bounded by the xy-plane, the xz-plane, the plane $x = y$ and the sphere of radius 3 about the origin. Use spherical coordinates to express the following triple integral as an iterated integral and evaluate the resulting iterated integral.

$$\iiint_R \frac{x}{x^2+y^2+z^2}\, dV$$



So the bounds are

$0 \le \rho \le 3$

$0 \le \theta \le \pi/4$

$0 \le \varphi \le \pi/2$

$x^2+y^2+z^2 = \rho^2$

$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$x = \rho \sin\varphi \cos\theta$

$\frac{x}{x^2+y^2+z} dV$

$= \rho \sin^2\varphi \cos\theta \, d\rho \, d\varphi \, d\theta$

So

$$\iiint_R \frac{x}{x^2+y^2+z^2} dV = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^3 \rho \sin^2\varphi \cos\theta \, d\rho \, d\varphi \, d\theta$$

separable integrand on rectangle in parameter-space <3

$$= \int_0^{\pi/2} \cos\theta \, d\theta \int_0^{\pi/4} \sin^2\varphi \, d\varphi \int_0^3 \rho \, d\rho$$

$$= \sin\theta \Big|_0^{\pi/2} \cdot \frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\Big|_0^{\pi/4} \cdot \frac{\rho^2}{2}\Big|_0^3$$

$$= (1-0)\cdot\left(\frac{\pi}{8} - \frac{1}{4} - (0-0)\right)\cdot\left(\frac{9}{2} - 0\right)$$

$$= \frac{9\pi}{16} - \frac{9}{8}$$

8. Find the average value of the function $w(x, y) = x^2y^2$ on the disk of radius 4 centered at the origin (in the xy-plane).

Area $= \pi \cdot 4^2 = 16\pi$

so avg $= \frac{1}{16\pi} \iint_R x^2y^2\, dA$

Handle it in polar coords:

region is $0 \le r \le 4$
$0 \le \theta \le 2\pi$

$x^2y^2 = r^2\cos^2\theta\, r^2\sin^2\theta$
$= r^4\cos^2\theta\sin^2\theta$

$dA = r\,dr\,d\theta$

so $\iint_R x^2y^2\, dA = \int_0^{2\pi}\int_0^4 r^5 \cos^2\theta\sin^2\theta\, dr\, d\theta$

separable integrand on rectangle in parameter-space

$= \int_0^{2\pi} \cos^2\theta \sin^2\theta\, d\theta \int_0^4 r^5\, dr$

$= \left.\frac{\theta}{8} - \frac{\sin 4\theta}{32}\right|_0^{2\pi} \cdot \left.\frac{r^6}{6}\right|_0^4$

$= \left[\frac{2\pi}{8} - 0 - (0-0)\right] \cdot \left[\frac{4^6}{6} - 0\right]$

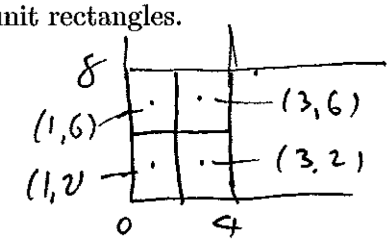
$= \frac{512\pi}{3}$

so the average is $\frac{512\pi}{3} \Big/ 16\pi = \frac{32}{3}$

9. (a) Let R be the rectangle $[0,4] \times [0,8]$ in the xy-plane. Give the midpoint Riemann sum approximation to the double integral

$$\iint_R x^2 + y \, dV$$

corresponding to the subdivision of R into four 2 by 4 unit rectangles.



So the Riemann sum is

$$(1^2+2)\cdot 2\cdot 4 + (3^2+2)\cdot 2\cdot 4$$
$$+ (1^2+6)\cdot 2\cdot 4 + (3^2+6)\cdot 2\cdot 4$$
$$= 8\,[5 + 11 + 7 + 15]$$
$$= 8\cdot 38 = 284$$

(b) Find the double integral of part (a) by iterated integration.

$$\int_0^8 \int_6^4 x^2+y \, dx\, dy = \int_0^8 \frac{x^3}{3} + yx \Big|_{x=0}^{x=4} dx$$

$$= \int_0^8 \frac{64}{3} + 4y \, dy = \frac{64}{3}y + 2y^2 \Big|_0^8$$

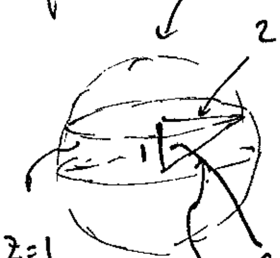
$$= \frac{512}{3} + 128$$

$$= \frac{512}{3} + \frac{384}{3} = \frac{896}{3}$$

10. Consider the triple integral

$$\iiint_R x^2 z \, dV$$

$\rho = \sqrt{3}$ $\quad z = \sqrt{3 - x^2 - y^2} = \sqrt{3 - r^2}$



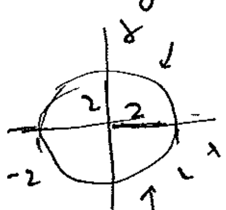
where R is the region lying inside the sphere of radius $\sqrt{3}$ about the origin and above the plane $z = 1$. For parts (a) through (c) use each of the indicated coordinate systems to give an iterated integral whose value is the given triple integral. Do not evaluate any of the iterated integrals. (Hints: remember the triangles whose trig functions you learned exact values for in high school trig, and in dealing with the plane $z = 1$ in spherical coordinates, solve for ρ.)

$z = 1$

$\rho \cos\varphi = 1$ $\quad \sqrt{3}$ $\quad \varphi = \pi/3$

$\rho = \sec\varphi$

proj ⊥



$y = \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$r = 2$

(a) Use rectangular coordinates

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{1}^{\sqrt{3-x^2-y^2}} x^2 z \, dz \, dy \, dx$$

(b) Use cylindrical coordinate

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{1}^{\sqrt{3-r^2}} r^2 \cos^2\theta \; z \; r \, dz \, dr \, d\theta$$

(c) Use spherical coordinates

$$\int_{0}^{2\pi} \int_{0}^{\pi/3} \int_{\sec\varphi}^{\sqrt{3}} \rho^5 \sin^3\varphi \cos\varphi \cdot \cos^2\theta \, d\rho \, d\varphi \, d\theta$$

cyl:
$x^2 = r^2 \cos^2\theta$
$dv = r \, dr \, d\theta$

spher:
$x^2 = \rho^2 \sin^2\varphi \cos^2\theta$
$dv = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$
$z = \rho \cos\varphi$

(d) Which of the three iterated integrals would you prefer to evaluate, and why? (The points for this part will be awarded for the coherence of your reason for your preference, not for which you choose.)

This is scored on the soundness of your reason.

I see good reasons for using cylindrical: the z-integral will result in the bounds being squared, leaving polynomials for the r-integral, so it's fairly easy.

or spherical: the ρ integral is easy and when its done you get an easy trigonometric integral since there's an odd power of $\sin\varphi$.

No matter what you do you'll have to integrate $\cos^2$ — even in rectangular after a trig substitution