Short answer questions (12 points each):

1. Recall that Green's Theorem can be used to express the area of a region as a line integral over the boundary of the region (Stewart gave three formulas, but any vector field for which the integrand in the double integral of Green's theorem equals the constant function 1 will work). Express the area inside one leaf of the three leafed rose given in polar coordinates by $r = \cos 3\theta$ as a line integral, and write the line integral as an ordinary integral by using the parametrization provided. Do not evaluate the resulting integral. The boundary of one leaf is given parametrically by

$$\vec{r}(t) = \cos(3t)\cos(t)\vec{\imath} + \cos(3t)\sin(t)\vec{\jmath} \quad t \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$$

Need $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

use $\langle 0, x \rangle$

So $\text{Area} = \int_{\partial R} x\,dy$

$x = \cos 3t \cos t$

$y = \cos 3t \sin t$

$dy = (-3\sin(3t)\sin t + \cos(3t)\cos t)\,dt$

$$A = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos(3t)\cos(t)\left[\cos(3t)\cos t - 3\sin(3t)\sin t\right] dt$$

2. The vector field on the plane $\vec{H}(x,y) = (y\cos(xy)+1))\vec{\imath} + (x\cos(xy)-1)\vec{\jmath}$ is conservative. Find a potential function $h$ such that $\nabla h = \vec{H}$.

Find $h$ by partial integration:

$h = \int y\cos(xy) + 1\,dx = y\sin(xy)\frac{1}{y} + x + C(y)$

$= \int x\cos(xy) - 1\,dy = x\sin(xy)\frac{1}{x} - y + k(x)$

$\therefore h(x,y) = \sin(xy) + x - y$ is a potential function

More short answer questions:

3. Express the flux of the vector field $\vec{H}(x, y, z) = x\vec{i} + y^2\vec{j} + z^3\vec{k}$ through the portion of the graph of $z = x^2 - y^2$ lying above (below, or on) the square $[0, 1] \times [0, 1]$ in the $xy$-plane as an ordinary iterated integral. Do not evaluate the iterated integral.

Parametrize the surface as a graph: $\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$

So $\vec{H}(\vec{r}(u,v)) = u\vec{i} + v^2\vec{j} + (u^2-v^2)^3\vec{k}$

$\vec{r}_u = \langle 1, 0, 2u \rangle$ $\qquad d\vec{S} = \vec{r}_u \times \vec{r}_v \, du\,dv = \langle -2u, 2v, 1 \rangle \, du\,dv$

$\vec{r}_v = \langle 0, 1, -2v \rangle$

$$\text{So flux} = \int_0^1 \int_0^1 -2u^2 + 2v^3 + (u^2 - v^2)^3 \, du\,dv$$

4. Express the average value of the function $f(x, y, z) = x + y^2 + z^3$ on the portion of the graph of $z = x^2 - y^2$ lying above (below, or on) the square $[0, 1] \times [0, 1]$ in the $xy$-plane as a ratio of ordinary iterated integrals. Do not evaluate (either of) the integrals.

from #3

$$dS = |d\vec{S}| = \sqrt{4u^2 + 4v^2 + 1}\, du\,dv$$

$$f(\vec{r}(u,v)) = u + v + (u^2 - v^2)^3$$

$$\text{Avg} = \frac{\int_0^1 \int_0^1 (u + v + (u^2 - v^2)^3)\sqrt{4u^2 + 4v^2 + 1}\, du\,dv}{\int_0^1 \int_0^1 \sqrt{4u^2 + 4v^2 + 1}\, du\,dv}$$

Yet more short answer questions:

5. Find the flux of the vector field $\vec{F}(x,y,z) = y\vec{i} - x\vec{j}$ through the sphere of radius 3 about the origin.

Use the divergence thm $\nabla \cdot \vec{F} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z} 0$

$= 0 + 0 + 0$

so flux $= \iiint\limits_R 0 \, dV = 0$

$R \leftarrow$ ball of radius 3 about $\vec{0}$

6. Give a vector equation for the plane passing through the point $(1, -3, \frac{1}{4})$ and perpendicular to the line $\vec{r}(t) = t\langle 2, -1, \frac{1}{4}\rangle + \langle 1, \frac{1}{2}, -1\rangle$.

So the direction vector of the line is normal to the plane in the eq n

$$\langle 2, -1, \tfrac{1}{4}\rangle \cdot \langle 1, -3, \tfrac{1}{4}\rangle = \langle 2, -1, \tfrac{1}{4}\rangle \cdot \langle x, y, z\rangle$$

7. Write, but do not solve, the system of (scalar) equations in the variables $x, y, z$, and $\lambda$ that you would need to solve in finding the maximum and minimum values of $f(x,y,z) = x + y^2 + z^3$ subject to the constraint $9x^2 + 4y^2 + z^2 = 36$ by the method of Lagrange multipliers.

$\nabla f = \lambda \nabla g$ and $9x^2 + 4y^2 + z^2 = 36$

$\langle 1, 2y, 3z\rangle = \lambda \langle 18x, 8y, 2z\rangle$

$$\boxed{\begin{aligned} 1 &= 18\lambda x \\ 2y &= 8\lambda y \\ 3z^2 &= 2\lambda z \\ 9x^2 + 4y^2 + z^2 &= 36 \end{aligned}}$$

Matching (20 points)

8. Ten formulas are given in the left column. The ten lettered items in the right column are descriptions or other formulas, each of which applies to exactly one item in the left column in the sense of correctly describing it or being equal to it. Put the letter of the item in the right column which applies to each item in the left column in the blank provided next to the corresponding item in the left column. Please uses block capital letters.

A $3\vec{j} + 2\vec{k}$

H $\iint_{\partial R} \langle x, -y, z \rangle \bullet d\vec{S}$

F $\iint_{\Sigma} \nabla \times \vec{F} \bullet d\vec{S}$

I $\cos(t)\vec{i} + \sqrt{t}\vec{j} + \sin(t)\vec{k}$

G $-y\vec{i} + x\vec{j} + \vec{k}$

J $\frac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{|\vec{r}_u(u,v) \times \vec{r}_v(u,v)|}$

E $x\vec{i} + y\vec{j} + \vec{k}$

D $\iint_{\Sigma} \vec{F} \bullet d\vec{S}$

C $\iint_{\Sigma} f\,ds$

B $\frac{\nabla f}{|\nabla f|}$

A. an irrotational vector field, which could also represent the flow of an incompressible fluid

B. gives unit normal vectors to level surfaces

C. useful in finding the average value of a function on a surface

D. useful in finding the net amount of fluid flowing through a surface in 3-space

E. an irrotational vector field which could not represent the flow of an incompressible fluid

F. $\int_{\partial \Sigma} \vec{F} \bullet d\vec{r}$

G. a vector field which is not irrotational but could represent the flow of an incompressible fluid

H. $\iiint_{R} dV$

I. a curve lying on the cylinder $x^2 + z^2 = 1$

J. gives unit normal vectors to a parametric surface

Long questions (26 points each)

9. Find the length of the curve given parametrically by

$$\vec{r}(t) = \langle 4\cos(t), 4\sin(t), 2t^2 \rangle \quad t \in [0, 2\pi]$$

$$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t), 4t \rangle$$

$$ds = |\vec{r}\,'(t)|\,dt = \sqrt{16\sin^2 t + 16\cos^2 t + 16t^2}\,dt$$

$$= 4\sqrt{1+t^2}\,dt$$

$$\text{So } \ell = \int_0^{2\pi} 4\sqrt{1+t^2}\,dt = 4\int_0^{2\pi} \sqrt{1+t^2}\,dt$$

by the integral table (5)

$$= 4\left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right|\right]\Bigg|_0^{2\pi}$$

$$= 4\left[\pi\sqrt{4\pi^2+1} + \frac{1}{2}\ln\left|2\pi+\sqrt{4\pi^2+1}\right|\right]$$

$$- 4\left[0 + \frac{1}{2}\ln\left|0+\sqrt{0^2+1}\right|\right]$$

$$= 4\pi\sqrt{4\pi^2+1} + 2\ln\left|2\pi+\sqrt{4\pi^2+1}\right|$$

10. Find the critical points of the function

$$f(x, y) = x^2y - \frac{1}{2}y^2 + \frac{5}{3}x^3 - \frac{3}{2}x^2$$

and use the second deriviative test to determine whether each is a local maximum, a local minimum or a saddle point.

at c.p.

$f_x(x,y) = 2xy + 5x^2 - 3x = 0$

$f_y(x,y) = x^2 - y = 0$

so at c.p.

$y = x^2$

$2x^3 + 5x^2 - 3x = 0$

$x(2x-1)(x+3) = 0$

$x = 0 \quad y = 0$

$x = \frac{1}{2} \quad y = \frac{1}{4}$

or $x = -3 \quad y = 9$

$f_{xx}(x,y) = 2y + 10x - 3$

$f_{xy}(x,y) = f_{yx}(x,y) = 2x$

$f_{yy}(x,y) = -1$

Hessian:

$$D = \begin{vmatrix} 2y+10x-3 & 2x \\ 2x & -1 \end{vmatrix} = -2y - 10x + 3 - 4x^2$$

$D(0,0) = 3 > 0 \qquad f_{yy}(0,0) = -1 < 0 \quad \therefore (0,0)$ is a local max

$D(\frac{1}{2}, \frac{1}{4}) = -2\cdot\frac{1}{4} - 10\cdot\frac{1}{2} + 3 - 4\cdot(\frac{1}{2})^2$

$= -\frac{1}{2} - 5 + 3 - 1 < 0 \qquad \therefore (\frac{1}{2}, \frac{1}{4})$ is a saddle pt

$D(-3, 9) = -2\cdot 9 - 10(-3) + 3 - 4(-3)^2$

$= -18 + 30 + 3 - 36 < 0 \qquad \therefore (-3, 9)$ is a saddle pt.

11. Recall that the helicoid is the surface given parametrically by

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$$

(a) Find the equation of the plane tangent to the helicoid at $(2, 2, \frac{\pi}{4})$.

at $(2, 2, \pi/4)$ $\quad v = \pi/4 \quad$ so $\quad 2 = u\cos(\pi/4) = u \cdot \sqrt{2}/2 \quad u = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

$$\vec{n} = \vec{r}_u \times \vec{r}_v(2\sqrt{2}, \pi/4)$$

$$= \frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j} + 2\sqrt{2}\vec{k}$$

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$

$\vec{r}_u \times \vec{r}_v = \sin v\,\vec{i} - \cos v\,\vec{j} + u\vec{k}$

So

$$\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\sqrt{2}\right\rangle \cdot \left\langle x-2, y-2, z-\pi/4\right\rangle = 0$$

is the tangent plane.

(b) Find the flux of the vector field $y\vec{i} - x\vec{j} - \vec{k}$ through the portion of the helicoid with $0 \le u \le 3$ and $0 \le v \le 2\pi$ (in the direction given by the normals determined by the parametrization as given).

$\vec{F}(\vec{r}(u,v)) = \langle u\sin v, -u\cos v, -1\rangle \qquad d\vec{S} = \langle \sin v, -\cos v, u\rangle\, du\,dv$

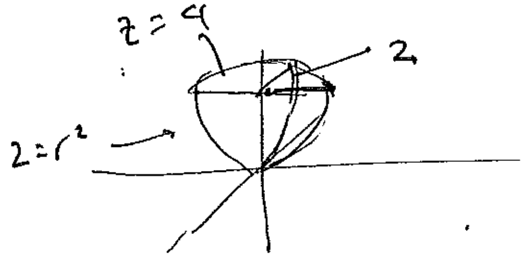
$$\text{Flux} = \iint_{\Sigma} \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 u\sin^2 v + u\cos^2 v - u \; du\,dv$$

$$= \int_0^{2\pi}\int_0^3 0 \; du\,dv = 0$$

12. Evaluate the triple integral

$$\iiint_R y\,dV$$

where $R$ is the region bounded by the circular paraboloid $z = x^2 + y^2$, the plane $z = 4$ and the $xz$-plane, which contains the point $(3, 1, 1)$. (Notice there are two bounded and four infinite regions bounded by those surfaces, so it is necessary to have some way of distinguishing them.)




Let's use cylindrical coords:

the region is bounded.

$r^2 \le z \le 4$

$0 \le r \le 2$

$0 \le \theta \le \pi$

$y = r\sin\theta$

$dV = r\,d\theta\,dz\,dr$

so $\iiint_R y\,dV = \int_0^2 \int_{r^2}^4 \int_0^\pi r\sin\theta \; r\,d\theta\,dz\,dr$

$= \int_0^2 \int_{r^2}^4 \int_0^\pi r^2 \sin\theta \, d\theta\,dz\,dr = \int_0^2 \int_{r^2}^4 -r^2\cos\theta \Big|_{\theta=0}^{\theta=\pi} dz\,dr$

$= \int_0^2 \int_{r^2}^4 2r^2\,dz\,dr = \int_0^2 2r^2(4 - r^2)\,dr$

$= \int_0^2 8r^2 - 2r^4\,dr = \frac{8}{3}r^3 - \frac{2}{5}r^5 \Big|_0^2$

$= \frac{64}{3} - \frac{64}{5} = \frac{128}{15}$