Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. You will have 1 hour and 50 minutes to complete this exam.

Question	Points	Score
1	15	
2	10	
3	20	
4	15	
5	10	
6	20	
7	10	
8	20	
9	15	
Total:	135	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. Let $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle 1, 1, 0\rangle$ and $\mathbf{w} = \langle 0, 1, 1\rangle$

   (a) (5 points) Compute the angle between $\mathbf{v} - \mathbf{w}$ and $\mathbf{u}$.

   (b) (5 points) Compute the volume of the parallelepiped $\mathcal{P}$ spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (c) (5 points) Give an equation for the plane orthogonal to $\mathbf{w}$ and passing through the point $(-1, 1, 1)$.

2. Consider the curve $\mathcal{C}$ parametrized by

$$\mathbf{r}(t) = \left\langle 2\sin(t), \sqrt{2}\cos(t), \sqrt{2}\cos(t) \right\rangle$$

for $0 \leq t \leq 2\pi$.

(a) (5 points) Find the unit tangent vector $\mathbf{T}(t)$ of $\mathbf{r}(t)$.

(b) (5 points) Calculate the length of $\mathcal{C}$.

3. Calculate the following quantities if they exist. Otherwise, explain why they do not exist.

   (a) (5 points)

   $$\lim_{(x,y)\to(0,0)} \frac{x^6 + y^6}{\sqrt{x^2 + y^2}}$$

   (b) (5 points) For $f(x, y) = ye^{\sqrt{x^2+y^2}}$, compute

   $$f_x(0, 0)$$

(c) (5 points) For $f(x, y, z) = x^2 - yz$ find a vector pointing in a direction where f increases, starting at $(2, 1, 3)$.

(d) (5 points) If $f(x, y, z) = \sin(xyz)$ and $\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right\rangle$, calculate $D_{\mathbf{u}}f(0, -1, \pi)$.

4. Let

$$f(x, y) = x^2 - yx + y^2$$

and

$$\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 8\}.$$

(a) (5 points) Find the critical points of $f(x, y)$ in $\mathcal{D}$.

(b) (5 points) Describe the local behavior of $f(x, y)$ at its critical points.

(c) (5 points) Find the global maximum and minimum values of $f(x, y)$ on $\mathcal{D}$.

5. (10 points) Let $\mathcal{W} = [0,1] \times [0,2] \times [0,3]$. Evaluate the triple integral

$$\iiint_{\mathcal{W}} x^3 y^2 z \, \mathrm{d}V$$

6. Let

$$F(x, y) = (3x + y, x + y).$$

(a) (5 points) Find the Jacobian of F and of its inverse function G.

(b) (5 points) Let

$$\mathcal{D} = \{(x, y) : 0 \leq 3x + y \leq 1, 0 \leq x + y \leq 1\},$$

and find its image in the uv-plane under F.

(c) (10 points) Calculate

$$\iint_{\mathcal{D}} xe^{6x+2y} + ye^{6x+2y} \ \mathrm{d}x \ \mathrm{d}y.$$

7. (10 points) Let $\mathcal{C}$ be the curve

$$\mathbf{r}(t) = \left\langle 2\sin(t), \sqrt{2}\cos(t), \sqrt{2}\cos(t) \right\rangle$$

for $0 \leq t \leq 2\pi$ which appeared in Problem 2. Let $\mathbf{F} = \langle y, z, x \rangle$ and compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \, d\mathbf{s}.$$

8. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle -y, x, z \rangle$$

and the region

$$\mathcal{W} = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, 0 \leq z\}$$

which is half of the unit ball.

(a) (5 points) Compute div($\mathbf{F}$) and directly evaluate

$$\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\ \mathrm{d}V.$$

(b) (5 points) The boundary of $\mathcal{W}$ consists of two smooth surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, oriented away from the interior. The first is the disc $\mathcal{S}_1 = \{(x, y, 0) : x^2 + y^2 \leq 1\}$. Evaluate

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot\ \mathrm{d}\mathbf{S}$$

(c) (5 points) The second surface is $\mathcal{S}_2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1, 0 \leq z\}$. Parameterize this surface and directly compute

$$\iint_{\mathcal{S}_2} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}$$

(d) (5 points) Write an equation involving your answers to (a), (b) and (c). What theorem explains their relationship?

9. Let

$$\mathbf{F} = \langle \cos(y), -x\sin(y) + z, y \rangle .$$

(a) (5 points) If $\mathbf{F}$ is a conservative vector field, find a potential. Otherwise, explain why it is not conservative.

(b) (5 points) Does $\mathbf{F}$ have a vector potential? Explain your response.

(c) (5 points) Suppose $\mathcal{C}$ is an oriented curve, starting at the origin $P = (0, 0, 0)$ and ending on the x-axis at $Q = (x, 0, 0)$. If

$$\int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = 42$$

then find Q.

Coordinate systems

Polar	Cylindrical	Spherical
$x = r\cos(\theta)$	$x = r\cos(\theta)$	$x = \rho\cos(\theta)\sin(\phi)$
$y = r\sin(\theta)$	$y = r\sin(\theta)$	$y = \rho\sin(\theta)\sin(\phi)$
	$z = z$	$z = \rho\cos(\phi)$
$r = \sqrt{x^2+y^2}$	$r = \sqrt{x^2+y^2}$	$\rho = \sqrt{x^2+y^2+z^2}$
$\theta = \tan^{-1}(y/x)$	$\theta = \tan^{-1}(y/x)$	$\theta = \tan^{-1}(y/x)$
	$z = z$	$\phi = \cot^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$
$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

Unit vectors

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \mathbf{N} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Useful area and volume formulas

$$\text{Surface area of sphere of radius } R = 4\pi R^2$$
$$\text{Volume of sphere of radius } R = \frac{4}{3}\pi R^3$$

Derivative formulas

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

Trig identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$$

Change of variables

$$G : \mathcal{D}_0 \to \mathcal{D}$$

$$G(u, v) = (x(u, v), y(u, v))$$

$$\text{Jac}(G) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint_{\mathcal{D}} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \iint_{\mathcal{D}_0} f(x(u, v), y(u, v))\ |\text{Jac}(G)|\ \mathrm{d}u\, \mathrm{d}v$$

Line integrals

$$\mathbf{c}(t) \text{ for } a \le t \le b \text{ parametrizing } \mathcal{C}$$

$$\int_{\mathcal{C}} f(x, y, z)\, \mathrm{d}s = \int_a^b f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, \mathrm{d}t$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, \mathrm{d}t$$

Surface integrals

$$G(u, v) = (x(u, v), y(u, v), z(u, v)) \text{ for } (u, v) \in \mathcal{D} \text{ parametrizing } \mathcal{S}$$

$$\mathbf{n}(u, v) = \mathbf{T}_u \times \mathbf{T}_v$$

$$\iint_{\mathcal{S}} f(x, y, z)\, \mathrm{d}S = \iint_{\mathcal{D}} f(G(u, v))\, \|\mathbf{n}(u, v)\|\, \mathrm{d}u\, \mathrm{d}v$$

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(u, v)) \cdot \mathbf{n}(u, v)\, \mathrm{d}u\, \mathrm{d}v$$