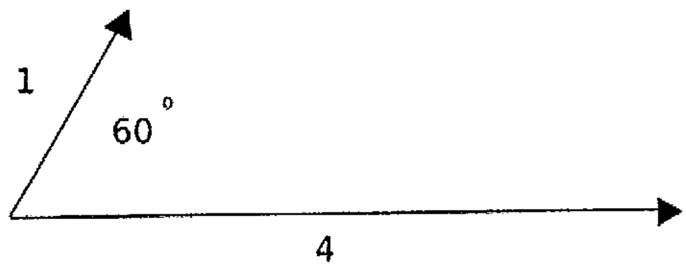
Short answer questions (6 points each):

Let the diagonal vector in the picture be denoted **u** and the horizontal vector be denoted **v**, with the magnitudes and angle between the vectors as indicated.




1. Find $|\mathbf{u} \times \mathbf{v}|$

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$
$$= 1 \cdot 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

2. Regarding the page as a plane in 3-space, does $\mathbf{u} \times \mathbf{v}$ point up out of the page toward you, or down toward the desk or table?

down

3. Find $\mathbf{u} \cdot \mathbf{v}$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$
$$= 1 \cdot 4 \cdot 1/2 = 2$$

Short answer questions, continued.

4. Give the coordinates in cylindrical coordinates and in spherical coordinates for the point given in rectangular coordinates by $(x, y, z) = (-3, 4, -5)$. You may express angular coordinates in terms of inverse trig functions.

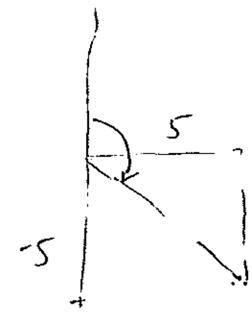
$r = \sqrt{(-3)^2 + 4^2} = 5$ $\quad (r, \theta, z) = (5, \pi - \arctan(4/3), -5)$

$x < 0$ so $\theta = \pi + \arctan(-4/3)$

$\rho = \sqrt{(-3)^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$

$\theta = \pi - \arctan(4/3)$

$\varphi = 3\pi/4$




$(\rho, \theta, \varphi) = (5\sqrt{2}, \pi - \arctan(4/3), \frac{3\pi}{4})$

5. Find $\langle 3, \frac{3}{2}, -1\rangle \times \langle 3, -2, 1\rangle$

$$\langle 3, 3/2, -1\rangle \times \langle 3, -2, 1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3/2 & -1 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= (3/2 - 2)\vec{i} - (3 - (-3))\vec{j} + (-6 - 9/2)\vec{k}$$

$$= -1/2\,\vec{i} - 6\vec{j} - \frac{21}{2}\vec{k} = \langle -1/2, -6, -21/2\rangle$$

Yet more short answer questions:

6. Find the angle between the vectors $\langle 3, \frac{3}{2}, -1\rangle$ and $\langle 3, -2, 1\rangle$. Your answer may involve inverse trig functions.

$\langle 3, 3/2, -1\rangle \cdot \langle 3, -2, 1\rangle = 9 - 3 - 1 = 5$

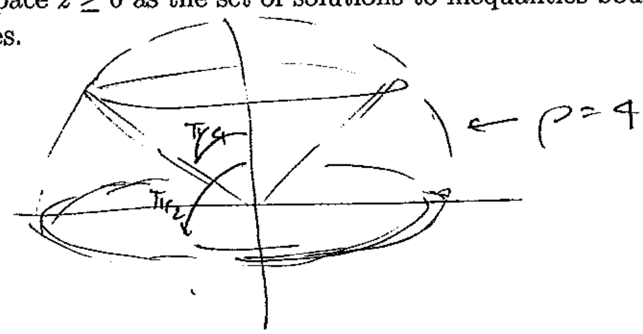
$\|\langle 3, 3/2, -1\rangle\| = \sqrt{9 + 9/4 + 1} = \sqrt{49/4} = 7/2$

$\|\langle 3, -2, 1\rangle\| = \sqrt{9 + 4 + 1} = \sqrt{14}$

So angle $\theta = \arccos \dfrac{5}{7/2 \cdot \sqrt{14}} = \arccos \dfrac{10\sqrt{14}}{98}$

$= \arccos \dfrac{10}{7\sqrt{14}} = \arccos \dfrac{5\sqrt{14}}{49}$

7. Describe the region lying inside a sphere of radius 4 about the origin, below the cone $z = \sqrt{x^2 + y^2}$ and in the half-space $z \geq 0$ as the set of solutions to inequalities bounding each of the spherical coordinates.




$0 \leq \rho \leq 4$

$\pi/4 \leq \varphi \leq \pi/2$

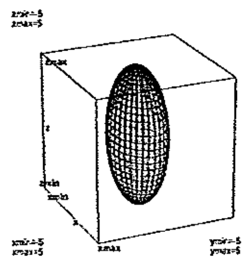
$0 \leq \theta \leq 2\pi$

8. For 3 points for each correct answer, identify the type of quadric surface represented by the picture or equation by writing the name (e.g. ellipsoid, hyperbolic parboloid, hyperboloid of two sheets) to the right of the picture or equation.

   (a) $x^2 - y^2/4 - z^2/9 = 1$ hyperboloid of two sheets

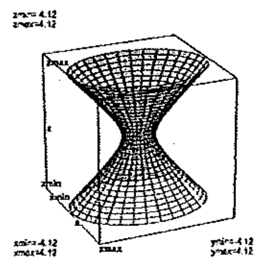
   (b) $z = 3y^2 - x^2$ hyperbolic paraboloid

   (c)

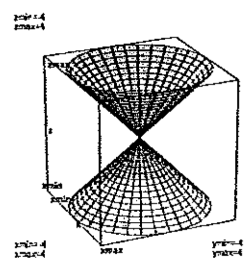
   ellipsoid

   (d)

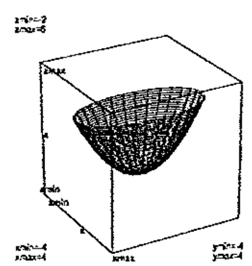
   hyperboloid of one sheet

   (e)

   (elliptic) cone

   (f)

   elliptic paraboloid

Long questions, each worth 20 points.

9. Consider the vector-valued function $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + \sqrt{t}\mathbf{j} + 3\sin(t)\mathbf{k}$ as a trajectory or equation of motion (i.e. giving the position of a particle at time $t$, say in seconds after some chosen time $t = 0$ as a displacement from the origin in some units, say m).

   (a) Find the velocity function.

$$\vec{v}(t) = \vec{r}'(t) = -3\sin(t)\vec{i} + \frac{1}{2}t^{-1/2}\vec{j} + 3\cos(t)\vec{k}$$

   (b) Find the acceleration function.

$$\vec{a}(t) = \vec{r}''(t) = -3\cos(t)\vec{i} - \frac{1}{4}t^{-3/2}\vec{j} - 3\sin(t)\vec{k}$$

   (c) Find the speed function.

$$v(t) = \|\vec{r}'(t)\| = \sqrt{9\sin^2 t + \frac{1}{4t} + 9\cos^2 t}$$

$$= \sqrt{9 + 1/4t} = \sqrt{\frac{36t+1}{4t}} = \frac{\sqrt{36t+1}}{2\sqrt{t}}$$

   (d) Find the function $\mathbf{T}(t)$ giving the unit tangent vector as a function of time.

$$\vec{T}(t) = \frac{1}{\frac{\sqrt{36t+1}}{2\sqrt{t}}} \left\langle -3\sin t, \tfrac{1}{2}t^{-1/2}, 3\cos(t) \right\rangle$$

$$= \frac{2\sqrt{t}}{\sqrt{36t+1}} \left\langle -3\sin t, \frac{1}{2\sqrt{t}}, 3\cos(t) \right\rangle$$

10. (a) Find a vector equation for the line passing through the points $(1, 0, 1)$ and $(-1, 2, 3)$.

Find a direction vector by taking the difference of the points

(as displacement vectors) $\vec{v} = \langle 1, 0, 1\rangle - \langle -1, 2, 3\rangle = \langle 2, -2, -2\rangle$

So the line has equation

$$\vec{r}(t) = \langle 1, 0, 1\rangle + t\langle 2, -2, -2\rangle$$

$$= \langle 2t+1, -2t, -2t+1\rangle$$

(b) Find an equation of the plane through the point $(1, 2, 3)$ perpendicular to the line in part (a).

The direction vector in (a) is a normal vector

to the plane, so the plane is given by

$$\langle 2, -2, -2\rangle \cdot \langle x, y, z\rangle = \langle 1, 2, 3\rangle \cdot \langle 2, -2, -2\rangle$$

$$2x - 2y - 2z = 2 - 4 - 6 = -8$$

$$2x - 2y - 2z = -8 \quad \text{or} \quad x - y - z = -4$$

11. A point mass of 1.00 kg is subject to a constant force of $\langle 1.00, 3.50, -2.00\rangle$ N. If at time $t = 0$ sec., the mass has a velocity of $\langle 1.00, 0.00, -1.00\rangle$ m/sec, and is located at the origin, find the equation of motion (i.e. position as a function of time).

(Hints to students who have not taken the relevant physics recently: N = kg·m/sec$^2$, and Newton's Second Law should really be state as a vector equation $\vec{F} = m\vec{a}$ (mass being a scalar, so the product on the right is scalar multiplication of a vector).)

So the acceleration is given by

$$\frac{1}{1.00\,\text{kg}}\langle 1.00, 3.50, -2.00\rangle\,\text{N} = \langle 1.00, 3.50, -2.00\rangle\,\text{m/sec}^2$$

∴ the velocity function

$$\vec{v}(t) = \int_0^t \langle 1.00, 3.50, -2.00\rangle\, d\tau + \langle 1.00, 0.00, -1.00\rangle \ \text{m/sec}$$

$$= t\langle 1.00, 3.50, -2.00\rangle + \langle 1.00, 0.00, -1.00\rangle \ \text{m/sec}$$

$$= \langle 1.00t + 1.00,\ 3.50t,\ -2.00t - 1.00\rangle \ \text{m/sec}$$

and the equation of motion is

$$\vec{r}(t) = \int_0^t \langle 1.00t + 1.00,\ 3.50t,\ -2.00t - 1.00\rangle\, dt + \langle 0.00, 0.00, 0.00\rangle$$

$$= \langle 0.500t^2 + 1.00t,\ 1.75t^2,\ -1.00t^2 - 1.00t\rangle \ \text{m}$$