Short answer questions.

1. (6) Find both first partial derivatives of $f(x, y) = y^x$ $= e^{x\ln(y)}$

$f_x(x,y) = e^{x\ln(y)} \cdot \ln(y) = \ln(y)\, y^x$

$f_y(x,y) = x y^{x-1}$

2. (9) Find the gradient of $g(x, y, z) = xyz - x^3$ at $(2, 1, -1)$

$\nabla g(x,y,z) = \langle yz - 3x^2, xz, xy \rangle$

$\nabla g(2,1,-1) = \langle 1\cdot(-1) - 3\cdot 2^2, 2(-1), 2\cdot 1 \rangle$

$= \langle -13, -2, 2 \rangle$

3. (9) Find the arc length of the curve $\mathbf{r}(t) = \langle 3\sin(t) - 1, -3\cos(t), 4t \rangle$ for $0 \leq t \leq 2$.

$\vec{r}'(t) = \langle 3\cos t, 3\sin t, 4 \rangle$

$\|\vec{r}'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$

$= \sqrt{9+16} = \sqrt{25} = 5$

so $\ell = \int_0^2 5\, dt = 10$

Short answer questions, continued.

4. (8) Find the indicated limit, or explain why it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+4y^2}$$

The limit does not exist:

Consider the [illegible] limits along lines $y = mx$

$$\lim_{x\to 0} \frac{x^2+(mx)^2}{x^2+4(mx)^2} = \lim_{x\to 0} \frac{x^2(1+m^2)}{x^2(1+4m^2)} = \frac{1+m^2}{1+4m^2}$$

which takes all values in $(\frac{1}{4}, 1]$

5. (8) Find the indicated limit, or explain why it does not exist:

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{x^2+4y^2}$$

(1,1) is in the domain of the rational function

So $\frac{x^2+y^2}{x^2+4y^2}$ is continuous at (1,1)

and

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{x^2+4y^2} = \frac{1^2+1^2}{1^2+4\cdot1^2} = 2/5$$

Yet more short answer questions.

6. (6) Set up, but do not evaluate, a definite integral whose value is the arc length of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Hint: the ellipse is given parametrically by $\mathbf{r}(t) = 5\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j}$ for $t \in [0, 2\pi)$.

$\vec{r}'(t) = -5\sin(t)\,\vec{i} + 4\cos(t)\,\vec{j}$

$\|\vec{r}'(t)\| = \sqrt{25\sin^2 t + 16\cos^2 t} = \sqrt{16 + 9\sin^2 t}$

So

$$\ell = \int_0^{2\pi} \sqrt{25\sin^2 t + 16\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{16 + 9\sin^2 t}\, dt$$

7. (6) Find the linearization of the function $f(x, y, z) = x\sqrt{y} + xz$ at the point $(1, 1, 2)$.

$f(1,1,2) = 1\sqrt{1} + 1\cdot 2 = 3$

$f_x(x,y,z) = \sqrt{y} + z \qquad f_x(1,1,2) = \sqrt{1} + 2 = 3$

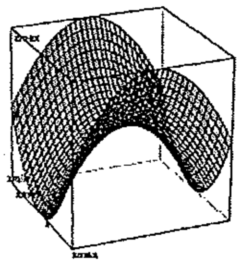
$f_y(x,y,z) = x\cdot\frac{1}{2}y^{-1/2} \qquad f_y(1,1,2) = 1/2$

$f_z(x,y,z) = x \qquad f_z(1,1,2) = 1$

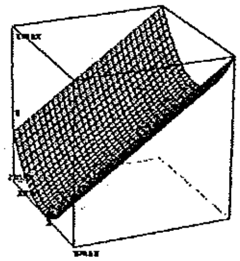
So the linearization is

$$L(x,y,z) = 3 + 3(x-1) + \tfrac{1}{2}(y-1) + 1\cdot(z-2)$$

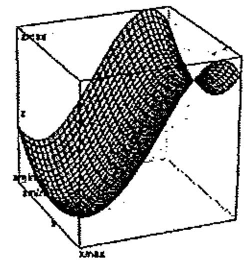
8. (12) Match the following graphs to the formula of the function of which they are the graph by putting the Roman numeral of the graph in the space provided next to the equation.



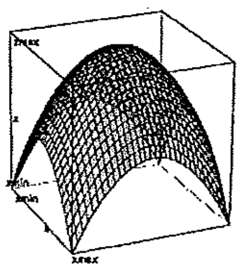
I.



II.



III.



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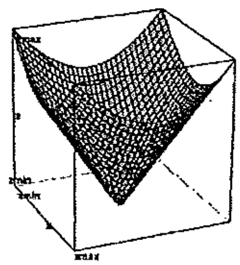
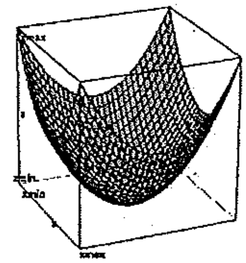


V.



VI.

VI $A(x,y) = 2x^2 + 2y^2$

I $B(x,y) = x^2 - y^2$

III $C(x,y) = x^2 - 2y^3 + 3y$

V $D(x,y) = \sqrt{x^2 + y^2}$

IV $E(x,y) = 4 - x^2 - y^2$

II $F(x,y) = x^2 + y$

9. (16) Find the curvature of the curve $\mathbf{f}(t) = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ at the point $(1, 1, 1)$. $= \vec{f}(1)$

$\vec{f}'(t) = \langle 2t, 1, 1 \rangle \qquad \vec{f}'(1) = \langle 2, 1, 1 \rangle$

$\vec{f}''(t) = \langle 2, 0, 0 \rangle \qquad \vec{f}''(1) = \langle 2, 0, 0 \rangle$

$\|\vec{f}'(1)\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$$\vec{f}'(1) \times \vec{f}''(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 0\vec{i} - (-2)\vec{j} + (-2)\vec{k}$$

$$= \langle 0, 2, -2 \rangle$$

$\|\vec{f}'(1) \times \vec{f}''(1)\| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$$\text{S.} \quad K(1) = \frac{2\sqrt{2}}{(\sqrt{6})^3} = \frac{2\sqrt{2}}{6\sqrt{6}} = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$$

10. (16) A particle is moving counter-clockwise around the circle of radius 5 about the origin with speed given by $v(t) = 1 - e^{-t}$.

(a) Find $a_{\mathrm{T}}(t)$, the magnitude of the tangential component of the acceleration as a function of time.

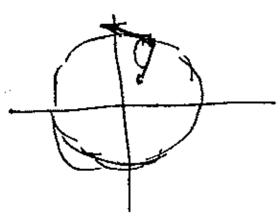
$$a_{\vec{T}}(t) = v'(t) = 0 - (-1)e^{-t} = e^{-t}$$

(b) Find $a_{\mathrm{N}}(t)$, the magnitude of the normal component of the acceleration as a function of time.

$$a_{\vec{N}}(t) = v(t)^2 \cdot K = (1-e^{-t})^2 \cdot \frac{1}{5}$$

$$= \frac{1}{5}(1-e^{-t})^2 = \frac{1}{5} - \frac{2}{5}e^{-t} + \frac{1}{5}e^{-2t}$$

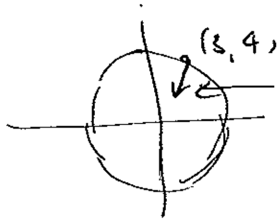
(c) When the particle is located at $(3,4)$ what is the unit tangent vector to its trajectory?



swapping coörd, and negating one from (d) gives

$$\vec{T} = \left(-\frac{4}{5}, \frac{3}{5}\right)$$

(d) When the particle is located at $(3,4)$ what is the (principal) unit normal vector to its trajectory?



unit normal points to origin

$$\therefore \text{ is } -\frac{(3,4)}{\|(3,4)\|} = \left(-\frac{3}{5}, -\frac{4}{5}\right)$$

11. (24) Consider the vector-valued function $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), \frac{3t}{2\pi} \rangle$ as a trajectory (i.e. giving the position of a particle at time t, say in seconds, after some chosen time $t = 0$ as a displacement from the origin in some units, say meters).

(a) Find the velocity, speed, and acceleration functions.

$\vec{v}(t) = \vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 3/2\pi \rangle$

$v(t) = \|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 9/4\pi^2}$

$\sqrt{4 + \frac{9}{4\pi^2}}$

$= \frac{1}{2\pi}\sqrt{16\pi^2+9}$

$\vec{a}(t) = \vec{r}\,''(t) = \langle -2\cos t, -2\sin t, 0 \rangle$

(b) Find the function giving the unit tangent vector to the trajectory at each point in time and its derivative.

$$\vec{T}(t) = \frac{2\pi}{\sqrt{16\pi^2+9}} \langle -2\sin t, 2\cos t, 3/2\pi \rangle$$

$$\vec{T}\,'(t) = \frac{2\pi}{\sqrt{16\pi^2+9}} \langle -2\cos t, -2\sin t, 0 \rangle$$

(c) Find the function giving the curvature of the trajectory at each point in time.

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin t & 2\cos t & 3/2\pi \\ -2\cos t & -2\sin t & 0 \end{vmatrix} = \langle \tfrac{3}{\pi}\sin t, -\tfrac{3}{\pi}\cos t, 4 \rangle$$

$$K = \frac{\frac{1}{\pi}\sqrt{16\pi^2+9}}{\left(\frac{1}{2\pi}\sqrt{16\pi^2+9}\right)^3} = \frac{8\pi^2}{(16\pi^2+9)}$$

$\|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{9/\pi^2 + 16} = 1/\pi \sqrt{16\pi^2+9}$

(d) Express the acceleration as the sum of two vectors, one $\mathbf{a}_T$ parallel to the velocity (whose magnitude gives the change in speed), the other $\mathbf{a}_N$ orthogonal to the velocity.

$$\vec{a}_{\vec{T}} = (\vec{a} \cdot \vec{T})\vec{T} = \left(\frac{2\pi}{\sqrt{16\pi^2+9}}\right)^2 \langle -2\cos t, -2\sin t, 0 \rangle \cdot \langle -2\sin t, 2\cos t, \tfrac{3}{2\pi} \rangle$$

or $$= \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \left(\frac{2\pi}{\sqrt{16\pi^2+9}}\right)^2 (4\cos t \sin t - 4\cos t \sin t + 0)$$

or $$= v'\vec{T} = 0$$

$\therefore\ \vec{a}_{\vec{N}} = \vec{a} = \langle -2\cos t, -2\sin t, 0 \rangle$