MATH 222 CALCULUS 3 SUMMER 2014: EXAM 1

Name: ____

Instructor: ______________________________________

To receive credit you must show your work.

Problem 1. The points $P = (2, -\sqrt{2}, -1)$, $Q = (1, 0, -2)$ and $R = (1, -\sqrt{2}, -1)$ determine a triangle in 3-space.

a)(10 points) Find out all three angles in degree.

Solution: $\vec{PQ} = Q - P = < -1, \sqrt{2}, -1 >$, $\vec{PR} = R - P = < -1, 0, 0 >$, $\vec{QR} = R - Q = < 0, -\sqrt{2}, 1 >$. Thus $cos\angle QPR = \frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}||\vec{PR}|} = \frac{1}{2}$, $cos\angle PQR = \frac{\vec{QP}\cdot\vec{QR}}{|\vec{QP}||\vec{QR}|} = \frac{\sqrt{3}}{2}$, $cos\angle QRP = \frac{\vec{RQ}\cdot\vec{RP}}{|\vec{RQ}||\vec{RP}|} = 0$. Therefore $\angle QPR = \frac{\pi}{3}$, $\angle PQR = \frac{\pi}{6}$, $\angle QRP = \frac{\pi}{2}$.

b)(10 points) Let $S = (0, 0, -2)$. Check whether these four points P, Q, R and S are in the same plane.

Solution: First find out the equation of the plane through P, Q and R. $\vec{\mathbf{n}} = \vec{PQ} \times \vec{PR} =< 0, 1, \sqrt{2} >$. Thus the equation is

$$y + \sqrt{2}z = -2\sqrt{2}.$$

Plug $S = (0, 0, -2)$ into the above equation. The equation holds. Then S is on the plane PQR. Thus these four points are in the same plane.

Problem 2. Compute the following:

(1) (10 points)

$$\frac{d}{dt}(< t^2, cos(t), 1 > \times < 0, ln(t), arctan(t) >);$$

Solution:

$$\frac{d}{dt}(< t^2, cos(t), 1 > \times < 0, ln(t), arctan(t) >)$$
$$= \frac{d}{dt} < cos(t)arctan(t) - ln(t), -t^2 arctan(t), t^2 ln(t) >$$
$$= < -sin(t)arctan(t) + \frac{cos(t)}{1+t^2} - \frac{1}{t}, -2tarctan(t) - \frac{t^2}{1+t^2}, 2tln(t) + t > .$$

(2) (10 points) Let $\mathbf{x}(t) =< -cos(t), sin(t), 1 >$ and $\mathbf{y}(t) =< -sin(t), cos(t), 0 >$:

$$\lim_{t\to 0} \|4\mathbf{x}(t) - 2\mathbf{y}(t)\|;$$

Solution: $\lim_{t\to 0} x(t) =< -1, 0, 1 >$, and $\lim_{t\to 0} y(t) =< 0, 1, 0 >$. Thus

$$\lim_{t\to 0} \|4\mathbf{x}(t) - 2\mathbf{y}(t)\| = \|4 < -1, 0, 1 > -2 < 0, 1, 0 > \| = \| < -4, -2, 4 > \| = 6.$$

Problem 3. (10 points) Find the intersection points of the surface $x = 2y^2 - 2z^2$ and the curve $\mathbf{r}(\vec{t}) = (4e^{t^2})\vec{\mathbf{i}} + (1 + \frac{e^{2t-1}}{2})\vec{\mathbf{j}} + (1 - \frac{e^{2t-1}}{2})\vec{\mathbf{k}}$.

Solution: Since $x = 2y^2 - 2z^2$ and $x = 4e^{t^2}$, $y = 1 + \frac{e^{2t-1}}{2}$, $z = 1 - \frac{e^{2t-1}}{2}$, we have

$$4e^{t^2} = 2(1 + \frac{e^{2t-1}}{2})^2 - 2(1 - \frac{e^{2t-1}}{2})^2.$$

The right hand side can be simplified to

$$\begin{aligned} 2(1 + \frac{e^{2t-1}}{2})^2 - 2(1 - \frac{e^{2t-1}}{2})^2 &= 2(1 + \frac{e^{4t-2}}{4} + e^{2t-1}) - 2(1 + \frac{e^{4t-2}}{4} - e^{2t-1}) \\ &= 4e^{2t-1}. \end{aligned}$$

Thus we have $4e^{t^2} = 4e^{2t-1}$, which implies $t^2 = 2t - 1$. Then $t = 1$. Therefore the intersection point is $< 4e, 1 + e/2, 1 - e/2 >$.

Problem 4. (10 points) Convert from cylindrical to spherical coordinates.

$$(2, \frac{\pi}{4}, 1).$$

Solution: First change from cylindrical coordinates to rectangle coordinates. $r = 2$, $t = \frac{\pi}{4}$, $z = 1$. Then $x = rcos(t) = 2cos(\frac{\pi}{4}) = \sqrt{2}$, $y = rsin(t) = \sqrt{2}$.

Then change from rectangle coordinates to spherical coordinates. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{5}$, $tan(\theta) = \frac{y}{x} = 1$, $cos(\phi) = \frac{z}{\rho} = \frac{1}{\sqrt{5}}$. So the spherical coordinates is $< \sqrt{5}, \frac{\pi}{4}, arccos\frac{1}{\sqrt{5}} >$.

Problem 5. Let $\vec{\mathbf{r}(t)} =< sin(t), -cos(t), t >$ be a curve in 3-space.

a)(10 points) Find out the equation for the tangent line of this curve at the point where $t = 0$.

Solution: $\vec{\mathbf{r}'(0)} =< 1, 0, 1 >$. $\vec{\mathbf{r}(0)} =< 0, -1, 0 >$. Thus the equation for the tangent line is $< x, y, z >=< 0, -1, 0 > +t < 1, 0, 1 >$.

b)(10 points) Find out the arc length function with the starting point where $t = 0$, and find out the arc length parametrization of the curve.

Solution:

$$s(t) = \int_0^t \|\vec{\mathbf{r}(t)}'\|dt = \int_0^t \sqrt{cos^2(t) + sin^2(t) + 1}dt = \sqrt{2}t.$$

Thus the arc length parametrization of the curve is $\vec{\mathbf{r_1}(s)} =< sin(\frac{s}{\sqrt{2}}), -cos(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}} >$

c)(10 points) Compute the curvature at the point $t = 0$.

Solution: $\vec{\mathbf{r}(0)}' =< cos(0), sin(0), 1 >=< 1, 0, 1 >$, $\vec{\mathbf{r}(0)}'' =< -sin(0), cos(0), 0 >=< 0, 1, 0 >$. Then $\kappa = \frac{\|\vec{\mathbf{r}(0)}' \times \vec{\mathbf{r}(0)}''\|}{\|\vec{\mathbf{r}(0)}'\|^3} = \frac{1}{2}$.

d)(10 points) Find the decomposition of $\vec{\mathbf{a}}(t)$ (which is the acceleration vector of $\vec{\mathbf{r}}(t)$) into tangential and normal components at $t = 0$.

Solution: $\vec{\mathbf{v}} = \vec{\mathbf{r}'(0)} =< 1, 0, 1 >$, $\vec{\mathbf{a}} = \vec{\mathbf{r}''(0)} =< 0, 1, 0 >$. $\vec{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{v}}||} =< \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} >$. $a_T = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = 0$. $\vec{\mathbf{a_N}} = \vec{\mathbf{a}} - \vec{\mathbf{T}} =< 0, 1, 0 >$. $a_N = ||\vec{\mathbf{a_N}}|| = 1$. $\vec{\mathbf{N}} = \frac{\vec{\mathbf{a_n}}}{a_N} =< 0, 1, 0 >$. Then $\vec{\mathbf{a}} = 0\vec{\mathbf{T}} + 1\vec{\mathbf{N}}$.