MATH 222 CALCULUS 3 SUMMER 2014: EXAM 2

Name: ____

Instructor: ____

To receive credit you must show your work.

Problem 1. (15 points) Compute

$$\lim_{(x,y)\to(0,0)} \frac{6x^2 - 3xy + y^2}{x^2 - 2y^2};$$

Solution: Along the path $y = x$, the limit is

$$\lim_{(x,y)\to(0,0)} \frac{6x^2 - 3x^2 + x^2}{x^2 - 2x^2} = \lim_{x\to 0} \frac{4x^2}{-x^2} = -4;$$

Along the path $y = 0$, the limit is

$$\lim_{(x,y)\to(0,0)} \frac{6x^2}{x^2} = 6;$$

Since $-4 \neq 6$, the original limit does not exist.

Problem 2. (10 points) Check whether the function

$$f(x,y) = \begin{cases} (3^{sin(\frac{ln(x^2+2\pi y^4+1)}{x^2+y^5})})^{y^3-2x} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

is continuous at $(0,0)$ or not.

Solution: Since $-1 \leq sin(\frac{ln(x^2+2\pi y^4+1)}{x^2+y^5}) \leq 1$, $\frac{1}{3}^{y^3-2x} \leq (3^{sin(\frac{ln(x^2+2\pi y^4+1)}{x^2+y^5})})^{y^3-2x} \leq 3^{y^3-2x}$. $\lim_{(x,y)\to(0,0)} \frac{1}{3}^{y^3-2x} = 1$, and $\lim_{(x,y)\to(0,0)} 3^{y^3-2x} = 1$. Thus from the squeeze theorem, $\lim_{(x,y)\to(0,0)}(3^{sin(\frac{ln(x^2+2\pi y^4+1)}{x^2+y^5})})^{y^3-2x} = 1$. Then from the definition of continuity, the function is continuous.

Problem 3. (10 points)Compute $\frac{\partial f}{\partial \phi}$ of $f(x, y, z) = xy + xz + yz$ at $(r, \phi, \theta) = (3, 0, 0)$, where $x = rsin(\phi)cos(\theta)$, $y = rsin(\phi)sin(\theta)$, $z = rcos(\phi)$.

Solution:$\frac{\partial f}{\partial x} = y + z$, $\frac{\partial f}{\partial y} = x + z$, $\frac{\partial f}{\partial z} = x + y$, $\frac{\partial x}{\partial \phi} = rcos(\phi)cos(\theta)$, $\frac{\partial y}{\partial \phi} = rcos(\phi)sin(\theta)$, $\frac{\partial z}{\partial \phi} = -rsin(\phi)$.

Thus

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$$

$$= (y + z)rcos(\phi)cos(\theta) + (x + z)rcos(\phi)sin(\theta) + (x + y)(-rsin(\phi))$$

$$= r^2(2sin(\phi)sin(\theta)cos(\phi)cos(\theta) + cos^2(\phi)cos(\theta) - sin^2(\phi)cos(\theta) + cos^2(\phi)sin(\theta) - sin^2(\phi)sin(\theta)).$$

At $(r, \phi, \theta) = (3, 0, 0)$, we have $\frac{\partial f}{\partial \phi}(3, 0, 0) = 9$.

Problem 4. a)(10 points) Find out the equation of the tangent space of the surface

$$z = -x^3 - 2xy + y^2$$

at $(x, y, z) = (1, 1, -2)$.

Solution: $z_x = -3x^2 - 2y$, $z_y = -2x + 2y$. Thus the equation of the tangent space is

$$z = z(1, 1) + z_x(1, 1)(x - 1) + z_y(1, 1)(y - 1) = -2 - 5(x - 1) = -5x + 3.$$

b)(10 points) Use linear approximation to approximate $z(1.02, 0.97)$ where $z(x, y) = -x^3 - 2xy + y^2$.

Solution: The linear approximation is $L(x, y) = -2 - 5(x - 1)$. Thus $z(1.02, 0.97) \approx -2 - 5(1.02 - 1) = -2.1$.

Problem 5. Let $f(x, y, z) = e^{x+y}z$.

a)(10 points) Compute the gradient of f.

Solution: $\nabla f =< f_x, f_y, f_z >=< e^{x+y}z, e^{x+y}z, e^{x+y} >$.

b)(10 points) Find the directional derivative of f at $(x, y, z) = (2, -2, 1)$ in the direction of the vector $< 3, 0, 4 >$.

Solution: $v =< 3, 0, 4 >$. $e_v = \frac{v}{||v||} =< \frac{3}{5}, 0, \frac{4}{5} >$.

$$D_{e_v} f = \nabla f \cdot e_v =< e^{2+(-2)}, e^{2+(-2)}, e^{2+(-2)} > \cdot < \frac{3}{5}, 0, \frac{4}{5} >=< 1, 1, 1 > \cdot < \frac{3}{5}, 0, \frac{4}{5} >= \frac{7}{5}.$$

Problem 6. Find out the extreme values of the function

$$f(x, y) = x^2 + 2y^2$$

inside the region $x^2 + y^2 = 1$ in the following steps.

a)(10 points) Find out all the critical points of f and use the second derivative test to determine the type of them.

Solution: $f_x = 2x$, $f_y = 4y$.

$$\begin{cases} 2x = 0, \\ 4y = 0. \end{cases}$$

So $x = 0$,$y = 0$. The critical point is $(0, 0)$.

The second derivative is $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 4$. Thus $D = f_{xx}f_{yy} - f_{xy}^2 = 8 > 0$. Thus the critical point is a local minimum.

b)(10 points)Using Lagrange multipliers to find the extreme values of f on the boundary $x^2 + y^2 = 1$.

Solution:

$$\begin{cases} 2x = \lambda(2x), \\ 4y = \lambda(2y), \\ x^2 + y^2 = 1. \end{cases}$$

The solutions are $x = 0, y = \pm 1$, or $x = \pm 1, y = 0$. $f(0, 1) = 2$, $f(0, -1) = 2$, $f(1, 0) = 1$, $f(-1, 0) = 1$. Thus the maximum on the boundary is 2, and the minimum on the boundary is 0.

c)(5 points) Find out the extreme values of the function f inside the region $x^2 + y^2 = 1$.
Solution: $f(0, 0) = 0$. Thus the global maximum is 2, and the global minimum is 0.