Midterm Exam I
Math 222 Summer 2015
June 19, 2015

Name:

Instructor's Name:

Problem(1) [18 points]: Let $\vec{v}$ = (2, -1, 1) and $\vec{w}$= (2, 6, 1). Compute the following:

a) Find the magnitude of ($4\vec{v}$-$\vec{w}$)

$4\vec{v}-\vec{w} = 4\langle 2,-1,1\rangle - \langle 2,6,1\rangle = \langle 8,-4,4\rangle - \langle 2,6,1\rangle$
$= \langle 6,-10,3\rangle$

$\|4\vec{v}-\vec{w}\| = \sqrt{(6)^2+(-10)^2+(3)^2}$
$= \sqrt{36+100+9} = \sqrt{145}$

b) Find the angle between the vectors $\vec{v}$ and $\vec{w}$

$\vec{v}\cdot\vec{w} = 4-6+1 = -1$

$\|\vec{v}\| = \sqrt{4+1+1} = \sqrt{6}$, $\|\vec{w}\| = \sqrt{4+36+1} = \sqrt{41}$

$\therefore \cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{-1}{\sqrt{6}\sqrt{41}} = \frac{-1}{\sqrt{246}}$

$\therefore \boxed{\theta = \cos^{-1}\left(\frac{-1}{\sqrt{246}}\right)}$

c) Find $\vec{v}$ x $\vec{w}$

$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 2 & 6 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ 6 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ 2 & 6 \end{vmatrix}$

$= \vec{i}(-1-6) - \vec{j}(2-2) + \vec{k}(12+2)$

$= -7\vec{i} + 0\vec{j} + 14\vec{k}$

$= \boxed{\langle -7, 0, 14\rangle}$.

Problem(2) [12 points]:
Find the area of the triangle with vertices P(1, 4, 6), Q(-2, 5, -1)and R(1, -1, 1).

$$\vec{PQ} = \langle -2, 5, -1\rangle - \langle 1, 4, 6\rangle = \langle -3, 1, -7\rangle$$

$$\vec{PR} = \langle 1, -1, 1\rangle - \langle 1, 4, 6\rangle = \langle 0, -5, -5\rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix}$$

$$= \vec{i}(-5-35) - \vec{j}\,|15+0| + \vec{k}\,|15-0|$$

$$= -40\vec{i} - 15\vec{j} + 15\vec{k} = \langle -40, -15, 15\rangle$$

$$\|\vec{PQ} \times \vec{PR}\| = \sqrt{1600 + 225 + 225}$$

$$= \sqrt{2050}$$

$\therefore$ Area of triangle PQR $= \frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$

$$= \frac{1}{2}\sqrt{2050}$$

$$= \boxed{\frac{5}{2}\sqrt{82}}$$

Problem(3) [12 points]:
Find the equation of the plane through the point (4, -2, 3) and parallel to the plane $3x - 7z = 12$.

we have $P_0(x_0, y_0, z_0) = (4, -2, 3)$

and normal vector of the plane $3x - 7z = 12$ is

$\vec{n} = \langle 3, 0, -7 \rangle$

Since the planes are parallel, the normal vector of required plane is $\vec{n} = \langle 3, 0, -7 \rangle$

Thus, Eqn of plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$3(x - 4) + 0(y + 2) + (-7)(z - 3) = 0$$

$$3x - 12 - 7z + 21 = 0$$

$$\boxed{3x - 7z + 9 = 0}$$

Problem(4) [10 points]:

Express each of the following sets in spherical coordinates:

i) $4 \leq x^2 + y^2 + z^2 \leq 16$.

$$\boxed{2 \leq \rho \leq 4}$$

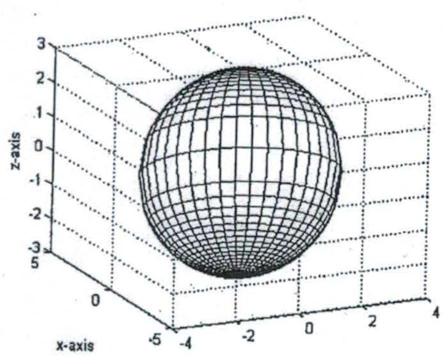
ii) $y \leq 0$

$$\boxed{\pi \leq \theta \leq 2\pi}$$

iii) $z \leq 0$

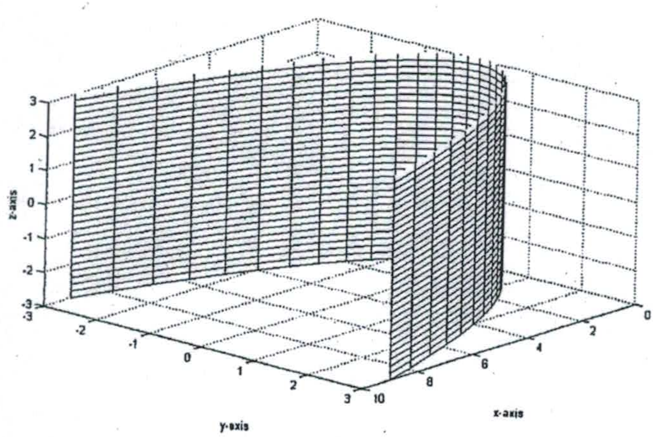
$$\boxed{\frac{\pi}{2} \leq \theta \leq \pi}$$

Problem(5) [12 points]:
Match ups. Here are some equations:
a) $x^2 + y^2 + z^2 = 9$
b) $z^2 - y^2 - x^2 = 4$
c) $z = 2x - 4y$
d) $x^2 - y^2 + z^2 = 0$
e) $x = y^2$
f) $x^2 + y^2 - z^2 = 16$
g) $z^2 - y^2 - x^2 = 0$
For each of the next four surfaces, determine which equation above determintes it.



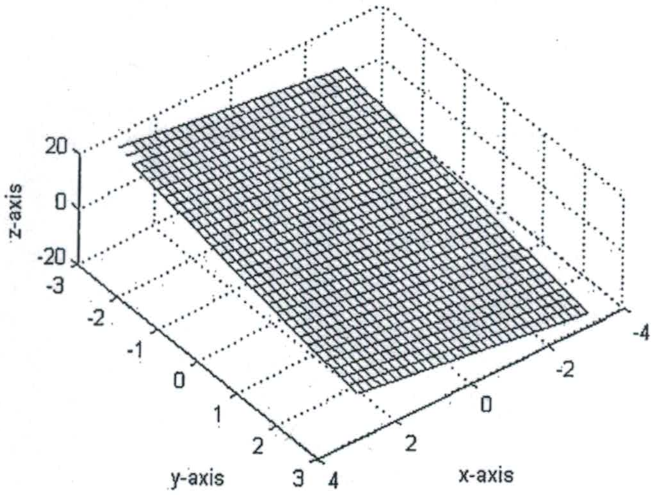
A

ⓐ $x^2+y^2+z^2=9$



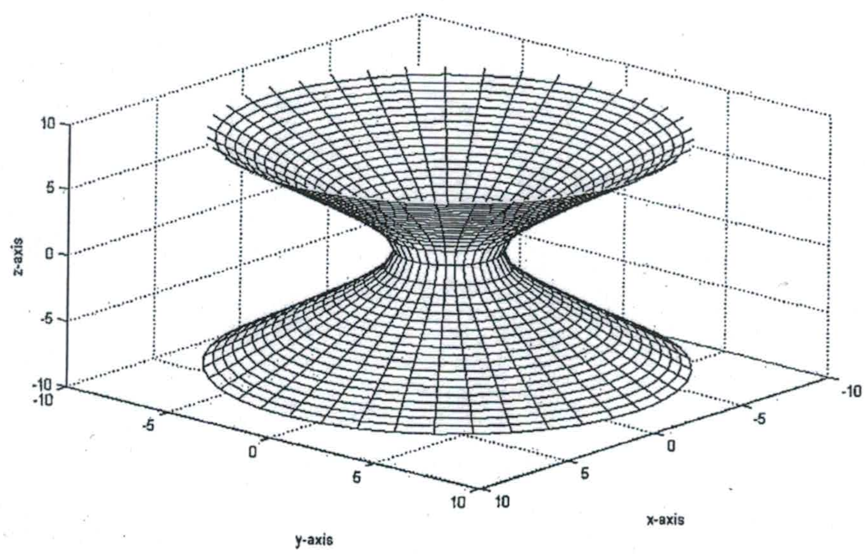
B

ⓔ $x = y^2$



C

ⓒ $z = 2x - 4y$



D

ⓕ $x^2 + y^2 - z^2 = 16$

or, $\boxed{x^2 + y^2 = z^2 + 16}$

Problem(6) [15 points]

Find the unit tangent , unit normal and binormal vectors for the circular helix: $r(\vec{t}) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$\therefore$ Unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1 \rangle$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0 \rangle$$

$$\|\vec{T}'(t)\| = \sqrt{\frac{1}{2}\left(\cos^2 t + \sin^2 t + 0\right)} = \frac{1}{\sqrt{2}}$$

unit normal vector $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|T'(t)\|} = \dfrac{\cancel{\frac{1}{\sqrt{2}}}\langle -\cos t, -\sin t, 0 \rangle}{\cancel{\frac{1}{\sqrt{2}}}}$

$$= \langle -\cos t, -\sin t, 0 \rangle$$

Binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\sin t & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{-\sin t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} \\ -\cos t & -\sin t \end{vmatrix}$$

$$= \vec{i}\left(0 + \frac{\sin t}{\sqrt{2}}\right) - \vec{j}\left(0 + \frac{\cos t}{\sqrt{2}}\right) + \vec{k}\left(\frac{\sin^2 t + \cos^2 t}{\sqrt{2}}\right)$$

$$= \boxed{\left\langle \frac{\sin t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$$

Problem(7) [15 points]:
Calculate the curvature of $r(\vec{t}) = (2sint, 1, 2cost)$

$\vec{r}'(t) = \langle 2\cos t, 0, -2\sin t \rangle$

$\vec{r}''(t) = \langle -2\sin t, 0, -2\cos t \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos t & 0 & -2\sin t \\ -2\sin t & 0 & -2\cos t \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 0 & -2\sin t \\ 0 & -2\cos t \end{vmatrix} - \vec{j} \begin{vmatrix} 2\cos t & -2\sin t \\ -2\sin t & -2\cos t \end{vmatrix} + \vec{k} \begin{vmatrix} 2\cos t & 0 \\ -2\sin t & 0 \end{vmatrix}$$

$$= \vec{i} \cdot 0 - \vec{j}(-4\cos^2 t - 4\sin^2 t) + \vec{k} \cdot 0$$

$$= 0\vec{i} + 4\vec{j} + 0\vec{k} = \langle 0, 4, 0 \rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{0 + 16 + 0} = \boxed{4}$$

$$\|\vec{r}'(t)\| = \sqrt{4\cos^2 t + 0 + 4\sin^2 t} = \boxed{2}$$

Thus, Curvature $K = \dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|r'(t)\|^3}$

$$= \frac{4}{2^3} = \frac{4}{8} = \boxed{\frac{1}{2}}$$

Problem(8) [6 points]: Short answer and multiple choice:

a) Let $\vec{v}$ and $\vec{w}$ be any two linear vectors. Which of the following is always true?.

i) $\vec{v}.\vec{w}$=0, ii) $\vec{v}$ x $\vec{w}$ =$\vec{0}$

$$\boxed{\vec{V} \times \vec{W} = \vec{0}}$$

b) Which has larger curvature, a circle with radius 40cm or a circle with radius 4cm?

A Circle with radius $\boxed{4\text{ cm}}$