MATH 222 SPRING 2017

EXAM 1

Problem 1.(14 pts) Find the projection of $\vec{v} = 2i + 3j + 4k$ onto $\vec{u} = i + j + k$.

$$\mathrm{Proj}_{\vec{u}}\,\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u};$$
$$\vec{v}\cdot\vec{u} = (2i+3j+4k)\cdot(i+j+k) = 2+3+4 = 9;$$
$$\|\vec{u}\|^2 = \|i+j+k\|^2 = 1^2+1^2+1^2 = 3;$$
$$\mathrm{Proj}_{\vec{u}}\,\vec{v} = \frac{9}{3}(i+j+k) = 3i+3j+3k.$$

Problem 2. (14 pts) Find the area of the triangle with the vertices $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,3)$.

$$\text{Area} = \frac{1}{2}\|\vec{AB}\times\vec{AC}\|;$$
$$\vec{AB} = -i+2j, \quad \vec{AC} = -i+3k;$$

$$\begin{aligned}\vec{AB}\times\vec{AC} &= \det\begin{bmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3\end{bmatrix} \\ &= \det\begin{bmatrix} 2 & 0 \\ 0 & 3\end{bmatrix} i - \det\begin{bmatrix} -1 & 0 \\ -1 & 3\end{bmatrix} j + \det\begin{bmatrix} -1 & 2 \\ -1 & 0\end{bmatrix} k \\ &= 6i+3j+2k;\end{aligned}$$

$$\text{Area} = \frac{\|6i+3j+2k\|}{2} = \frac{\sqrt{36+9+4}}{2} = \frac{7}{2}.$$

Problem 3. (14 pts) Find the angle between the vectors $\vec{u} = i - 2j + 2k$ and $\vec{v} = 6i + 3j + 2k$. Give the answer in terms of inverse trigonometric functions.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|};$$
$$\vec{u}\cdot\vec{v} = (i - 2j + 2k)\cdot(6i + 3j + 2k) = 6 - 6 + 4 = 4;$$
$$\|\vec{u}\| = \|i - 2j + 2k\| = \sqrt{1+4+4} = 3;$$
$$\|\vec{v}\| = \|6i + 3j + 2k\| = \sqrt{36+9+4} = 7;$$
$$\cos\theta = 4/21, \quad \theta = \arccos(4/21).$$

Problem 4. (14 pts) Find an equation of the plane passing through the points $A(1,1,1)$, $B(1,2,3)$ and $C(2,1,2)$.

The equation is of the form

$$ax + by + cz + d = 0,$$

where

$$\vec{n} = ai + bj + ck$$

is a normal vector. One may find $\vec{n}$ as

$$\begin{aligned}\vec{n} = \vec{AB}\times\vec{AC} = (j + 2k)\times(i + k) &= j\times i + j\times k + 2k\times i + 2k\times k\\ &= -k + i + 2j + 0 = i + 2j - k.\end{aligned}$$

Thus the equation of the plane can be written as

$$x + 2y - z + d = 0.$$

Plug in the coordinates of the point A to obtain

$$1 + 2 - 1 + d = 0, \quad d = -2,$$

and hence the equation is

$$x + 2y - z - 2 = 0.$$

Problem 5. (14 pts) Find the distance from the origin $O(0,0,0)$ to the line L given by the parametric equations

$$\begin{cases} x = 1 + t, \\ y = 2 + t, \\ z = 3 + t. \end{cases}$$

Observe that $A(1,2,3)$ is a point on the line and $\vec{v} = i + j + k$ gives a direction of the line. One has

$$\operatorname{dist}(O, L) = \frac{\|\vec{OA} \times \vec{v}\|}{\|\vec{v}\|};$$

$$\vec{OA} \times \vec{v} = (i + 2j + 3k) \times (i + j + k) = \det \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} i - \det \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} j + \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} k = -i + 2j - k;$$

$$\|\vec{OA} \times \vec{v}\| = \| - i + 2j - k\| = \sqrt{6}, \quad \|\vec{v}\| = \|i + j + k\| = \sqrt{3},$$

$$\operatorname{dist}(O, L) = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}.$$

Problem 6. (14 pts) Find the distance from the origin $O(0,0,0)$ to the plane P given by the equation

$$x + 2y + 2z + 1 = 0.$$

Observe that $A(-1,0,0)$ is a point in the plane and $\vec{n} = i + 2j + 2k$ is a normal vector for the plane. One has

$$\operatorname{dist}(O, P) = \frac{|\vec{OA} \cdot \vec{n}|}{\|\vec{n}\|};$$

$$\vec{OA} \cdot \vec{n} = -i \cdot (i + 2j + 2k) = -1;$$

$$\|\vec{n}\| = \|i + 2j + 2k\| = \sqrt{9} = 3;$$

$$\operatorname{dist}(O, P) = \frac{1}{3}.$$

Problem 7. (16 pts) A quadric surface is given in cylindrical coodinates (r, θ, z) by the equation

$$z = r^2(1 + \sin^2\theta).$$

Write an equation of the surface in rectangular coordinates (x, y, z). Classify the surface (as an ellipsoid, a hyperboloid or a paraboloid).

Re-write the equation using the formulas $r^2 = x^2 + y^2$, $y = r\sin\theta$:

$$z = r^2(1 + \sin^2\theta) = r^2 + (r\sin\theta)^2 = x^2 + y^2 + y^2 = x^2 + 2y^2.$$

Consider the traces of the surface

$$z = x^2 + 2y^2$$

in the planes perpendicular to the axes:

- In the plane $x =$ const the trace is
  $$z = 2y^2 + \text{const}.$$
  This is a parabola.
- In the plane $y =$ const the trace is
  $$z = x^2 + \text{const}.$$
  This is a parabola.
- In the plane $z =$ const the trace is
  $$x^2 + 2y^2 = \text{const}.$$
  This is an ellipse if the constant is nonnegative; the trace is empty otherwise.

Thus the surface is an elliptic paraboloid.