MATH 222 SPRING 2017

EXAM 2

Problem 1.(20 pts) Let $f(x,y) = x^3 + 2xy + 3y^2$.
a) Find an equation of the plane tangent to the surface $z = f(x,y)$ at the point $(1,-1,2)$.

The tangent plane has an equation of the form

$$z - 2 = f_x(1,-1)(x-1) + f_y(1,-1)(y+1).$$

Since

$$f_x(1,-1) = \frac{\partial f}{\partial x}\Big|_{\substack{x=1\\ y=-1}} = (3x^2+2y)\Big|_{\substack{x=1\\ y=-1}} = 1,$$

$$f_y(1,-1) = \frac{\partial f}{\partial y}\Big|_{\substack{x=1\\ y=-1}} = (2x+6y)\Big|_{\substack{x=1\\ y=-1}} = -4,$$

the tangent plane is given by

$$z - 2 = (x-1) - 4(y+1).$$

b) Use part a) to estimate $f(1.01, -1.02)$.

Using the equation of the tangent plane for linear approximation,

$$f(1.01,-1.02) - 2 \approx (1.01-1) - 4(-1.02+1),$$

hence

$$f(1.01,-1.02) \approx 2 + 0.01 + 0.08 = 2.09.$$

Problem 2. (20 pts) Let $f(x,y) = x^3 + 3xy - 2y^2$.
a) Find the directional derivative of $f(x,y)$ at the point $(1,1)$ in the direction of the vector $v = 4i - 3j$.

The directional derivative is given by

$$D_u f(1,1) = \nabla f(1,1) \cdot u,$$

where

$$u = \frac{v}{\|v\|} = \frac{4i - 3j}{5}$$

is the direction of the vector v and

$$\nabla f(1,1) = f_x(1,1)i + f_y(1,1)j$$

is the gradient of $f(x,y)$ at the point $(1,1)$. Since

$$f_x(1,1) = \frac{\partial f}{\partial x}\bigg|_{\substack{x=1\\ y=1}} = (3x^2+3y)\bigg|_{\substack{x=1\\ y=1}} = 6,$$

$$f_y(1,1) = \frac{\partial f}{\partial y}\bigg|_{\substack{x=1\\ y=1}} = (3x-4y)\bigg|_{\substack{x=1\\ y=1}} = -1,$$

one has

$$\nabla f(1,1) = 6i - j$$

and

$$D_u f(1,1) = (6i - j) \cdot (\frac{4}{5}i - \frac{3}{5}j) = \frac{27}{5}.$$

b) Find the direction of the most rapid increase of $f(x,y)$ at the point $(1,1)$ and the directional derivative in that direction.

The direction of the most rapid increase of $f(x,y)$ at the point $(1,1)$ is the direction of the gradient:

$$u_0 = \frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = \frac{6i - j}{\sqrt{37}}.$$

The directional derivative in the direction u_0 is:

$$D_{u_0} f(1,1) = \nabla f(1,1) \cdot u_0 = \|\nabla f(1,1)\| = \sqrt{37}.$$

Problem 3. (20 pts) Let function $z = f(x, y)$ be given implicitly by the equation

$$x^3 + xz = z^3 + 2yz.$$

Find the gradient of $f(x, y)$ at the point $(2, 1)$.

Re-write the equation for z in the form $g(x, y, z) = 0$, where

$$g(x, y, z) = x^3 + xz - 2yz - z^3.$$

The chain rule implies that

$$\frac{\partial z}{\partial x} = -\left(\frac{\partial g}{\partial x}\right) \Big/ \left(\frac{\partial g}{\partial z}\right) = -\frac{3x^2 + z}{x - 2y - 3z^2}.$$

Note that for $(x, y) = (2, 1)$ z is given by $8 + 2z = z^3 + 2z$, hence $z^3 = 8$ and $z = f(2, 1) = 2$. Therefore,

$$f_x(2, 1) = \left.\frac{\partial z}{\partial x}\right|_{\substack{x=2\\ y=1\\ z=2}} = \frac{14}{12} = \frac{7}{6}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\left(\frac{\partial g}{\partial y}\right) \Big/ \left(\frac{\partial g}{\partial z}\right) = \frac{2z}{x - 2y - 3z^2}$$

and

$$f_y(2, 1) = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=2\\ y=1\\ z=2}} = -\frac{4}{12} = -\frac{1}{3}.$$

Thus

$$\nabla f(2, 1) = f_x(2, 1)i + f_y(2, 1)j = \frac{7}{6}i - \frac{1}{3}j.$$

Problem 4. (20 pts) Find and classify (as a local maximum, a local minimum or a saddle) the critical points of the function

$$f(x, y) = x^3 - 6xy + y^2.$$

The critical points of $f(x, y)$ are solutions of the equation

$$\nabla f(x, y) = 0.$$

Since

$$\frac{\partial f}{\partial x} = 3x^2 - 6y, \quad \frac{\partial f}{\partial y} = -6x + 2y,$$

the critical pints are given by the system of equations

$$\begin{cases} 3x^2 - 6y = 0, \\ -6x + 2y = 0. \end{cases}$$

The second equation implies $y = 3x$ which can be substituted in the first equation to yield $3x^2 - 18x = 0$. Thus $x = 0$ or $x = 6$ and the critical points are $(x, y) = (0, 0)$ and $(x, y) = (6, 18)$.

To classify these points, compute the second order derivatives

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -6, \quad \frac{\partial^2 f}{\partial y^2} = 2,$$

and form the discriminant

$$D(x, y) = \det \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix} = \det \begin{bmatrix} 6x & -6 \\ -6 & 2 \end{bmatrix} = 12x - 36.$$

Since $D(0, 0) = -36 < 0$, the critical point $(0, 0)$ is a saddle point. Since $D(6, 18) = 36 > 0$ and $f_{xx}(6, 18) = 36 > 0$, the critical point $(6, 18)$ is a point of local minimum.

Problem 5. (20 pts) Find the maximum and minimum values of the function $f(x, y) = 3x^2 + 4xy$, subject to constraint $x^2 + y^2 = 5$.

The maximum and minimum values are attained at the critical points (x, y) which satisfy the system of equations

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y), \\ f_y(x, y) = \lambda g_y(x, y), \\ \ g(x, y) = 0. \end{cases}$$

Here $g(x, y) = x^2+y^2-5$ is the constraint function and λ - the Lagrange multiplier - is an unknown constant. Calculating the partial derivatives of $f(x, y)$ and $g(x, y)$, one obtains the following system:

$$\begin{cases} 6x + 4y = 2\lambda x, \\ \qquad 4x = 2\lambda y, \\ x^2 + y^2 = 5, \end{cases}$$

which can be transformed by substitution into

$$\begin{cases} 3\lambda y + 4y = \lambda^2 y, \\ \qquad\quad x = (\lambda/2)y, \\ x^2 + y^2 = 5, \end{cases}$$

The first equation implies that

$$(\lambda^2 - 3\lambda - 4)y = 0.$$

Note that $y \neq 0$ (otherwise the second equation implies that $x = y = 0$ which, in view of the third equation, is impossible). Thus $\lambda^2-3\lambda-4 = 0$ and $\lambda = 4$ or $\lambda = -1$.

If $\lambda = 4$ then the second equation yields $x = 2y$ and the last equation of the system becomes $5y^2 = 5$. Thus in this case $y = \pm 1$, which leads to the critical points $(2, 1)$ and $(-2, -1)$ where $f(x, y)$ attains the value

$$f(2, 1) = f(-2, -1) = 20.$$

If $\lambda = -1$ then the second equation yields $x = (-y)/2$ and the last equation of the system becomes $5y^2/4 = 5$. Thus in this case $y = \pm 2$, which leads to the critical points $(-1, 2)$ and $(1, -2)$ where $f(x, y)$ attains the value

$$f(-1, 2) = f(1, -2) = -5.$$

To summarize, the maximum and minimum values of the function $f(x, y) = 3x^2 + 4xy$, subject to constraint $x^2 + y^2 = 5$, are 20 and -5, respectively.