Problem 1. (20 pts) Let L denote the line given by the parametric equation

$$r(t) = (2+2t)i + (3+t)j + (1+2t)k.$$

a) Find the distance from L to the origin.

Note that L contains the point $P(2,3,1)$ and has a direction of

$$\vec{v} = 2i + j + 2k.$$

Hence

$$\operatorname{dist}(O,L) = \frac{\|\vec{OP} \times \vec{v}\|}{\|v\|} = \frac{1}{3}\left\|\det \begin{bmatrix} i & j & k \\ 2 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}\right\| = \frac{\|5i - 2j - 4k\|}{3} = \sqrt{5}.$$

b) Find an equation of the plane which contains both the origin and the line L.

A normal to the plane would be perpendicular to both $\vec{OP}$ and $\vec{v}$, therefore it has a direction of

$$\vec{OP} \times \vec{v} = 5i - 2j - 4k$$

and an equation of the form

$$5x - 2y - 4z = d.$$

Since the plane contains the origin, $d = 0$ and the equation is

$$5x - 2y - 4z = 0.$$

Problem 2. (20 pts) Let Γ denote the curve given by the parametric equation

$$r(t) = 3ti + 4\cos(t)j + 4\sin(t)k, \quad 0 \le t \le 1.$$

a) Find the length of Γ.

$$\begin{aligned}\text{length}(\Gamma) = \int_\Gamma ds = \int_0^1 \|r'(t)\|dt = \int_0^1 \|3i - 4\sin(t)j + 4\cos(t)k\|dt \\ = \int_0^1 \sqrt{9 + 16(\sin^2 t + \cos^2 t)}dt = \int_0^1 5dt = 5.\end{aligned}$$

b) Find the curvature of Γ.

Begin with the unit tangent vector:

$$T(t) = \frac{dr}{ds} = \frac{r'(t)}{\|r'(t)\|} = \frac{3i - 4\sin(t)j + 4\cos(t)k}{5}.$$

Now the curvature is

$$\kappa = \left\|\frac{dT}{ds}\right\| = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\| - 4\cos(t)j - 4\sin(t)k\|}{25} = \frac{4}{25}.$$

Problem 3. (20 pts) Let D denote the closed triangular domain

$$D = \{(x, y) \,|\, x \geq 0, y \geq 0, 2x + y \leq 2\}.$$

Find the maximal and minimal values of the function

$$f(x, y) = x^2 + 2xy - y^2$$

in the domain D.

First, find the critical points of $f(x, y)$:

$$\begin{cases} \partial f/\partial x = 0, \\ \partial f/\partial y = 0, \end{cases} \quad \begin{cases} 2x + 2y = 0, \\ 2x - 2y = 0, \end{cases} \quad \begin{cases} x = 0, \\ y = 0. \end{cases}$$

Thus $f(x, y)$ has no critical points *interior* to D and the maximal and minimal values are attained on the boundary, which consists of three line segments:

$$I : 0 \leq x \leq 1, y = 0;$$
$$II : x = 0, 0 \leq y \leq 2;$$
$$III : 2x + y = 2, x \geq 0, y \geq 0.$$

On segment I the function is $f(x, 0) = x^2$; its maximal and minimal values are

$$f(1, 0) = 1, \quad f(0, 0) = 0.$$

On segment II the function is $f(0, y) = -y^2$; its maximal and minimal values are

$$f(0, 0) = 0, \quad f(0, 2) = -4.$$

On segment III the function $f(x, y) = x^2 + 2xy - y^2$ is subject to constraint

$$2x + y = 2.$$

It will attain the maximal and minimal values either at the endpoints (which have already been considered above) or at a critical point where

$$\nabla f = \lambda \nabla (2x + y - 2).$$

This leads to a system of equatiions

$$\begin{cases} 2x + 2y = 2\lambda, \\ 2x - 2y = \lambda, \\ 2x + y = 2, \end{cases} \quad \begin{cases} \lambda = x + y, \\ x = 3y, \\ 7y = 2, \end{cases}$$

so the critical point is at $(x, y) = (6/7, 2/7)$. Since this critical point is in the first quadrant, it belongs to D; the corresponding value of the function is $f(6/7, 2/7) = 8/7$. By comparison, the maximal and minimal values of $f(x, y)$ in D are $8/7$ and -4, respectively.

Problem 4. (20 pts)
a) Compute the integral

$$\iint_D (x^2+y^2)^3 dxdy,$$

where D is the unit disk centered at the origin.
Using polar coordinates:

$$\iint_D (x^2+y^2)^3 dxdy = \iint_D (r^2)^3 r dr d\theta$$
$$= \int_0^{2\pi}\int_0^1 r^7 dr d\theta = \int_0^{2\pi} \frac{d\theta}{8} = \frac{\pi}{4}.$$

b) Use Green's theorem to compute the line integral

$$\oint_\Gamma (\cos(x^2)+e^y)dx + (e^x+\sin(y^2))dy,$$

where Γ is the square with the verices $(0,0),(0,10),(10,10),(10,0)$.

$$\oint_\Gamma (\cos(x^2)+e^y)dx + (e^x+\sin(y^2))dy$$
$$= \int_0^{10}\int_0^{10} \left(\frac{\partial(e^x+\sin(y^2))}{\partial x} - \frac{\partial(\cos(x^2)+e^y)}{\partial y}\right) dxdy$$
$$= \int_0^{10}\int_0^{10} (e^x - e^y)\, dxdy = 0.$$

Problem 5. (20 pts) Is the vector field

$$F(x, y, z) = (2x + yz)i + (2y + xz)j + (2z + xy)k$$

conservative? If yes, find a function $f(x, y, z)$ such that $\nabla f = F$.

Note that

$$\nabla \times F = \det \begin{bmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + yz & 2y + xz & 2z + xy \end{bmatrix}$$
$$= (x - x)i - (y - y)j + (z - z)k = 0$$

in a simply connected domain (the whole space $\mathbb{R}^3$). Hence F is conservative.

To find a potential function f, solve the system

$$\begin{cases} \partial f/\partial x = 2x + yz, \\ \partial f/\partial y = 2y + xz, \\ \partial f/\partial z = 2z + xy. \end{cases}$$

Integrating each of the three equations leads to

$$f(x, y, z) = x^2 + xyz + \varphi_1(y, z) = y^2 + xyz + \varphi_2(x, z) = z^2 + xyz + \varphi_3(x, y).$$

Comparing the three expressions, one concludes that

$$f(x, y, z) = x^2 + y^2 + z^2 + xyz \, (+ \text{ an arbitrary constant}).$$