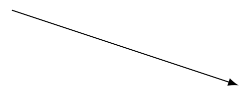
1. Here is a vector which you can assume has unit length:



Call this vector **u**. Now using the same base point draw a vector **w** (and label it) so that the following are all satisfied:

(a) $|\mathbf{w}| = 1$.

(b) $\mathbf{u} \times \mathbf{w}$ points toward you

(c) $\mathbf{u} \times \mathbf{w} \approx \sqrt{3}/2$. (Try to make it as close as you can.)

Next using the same base point again draw a vector **v** (and label it) so that the following are all satisfied:

(a) $|\mathbf{v}| = 1$.

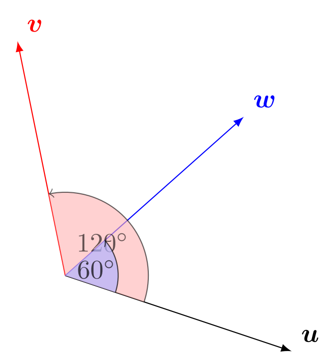
(b) $\mathbf{u} \cdot \mathbf{v} \approx -1/2$. (Again, do your best to get equality.)

(c) $\mathbf{u} \times \mathbf{v}$ points toward you.

Solution:

$$\mathbf{u} \times \mathbf{w} = \|\mathbf{u}\|\|\mathbf{w}\| \sin\theta = \sqrt{3}/2 \implies \theta = 60^\circ$$
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = -1/2 \implies \theta = 120^\circ$$

so the final configuration looks like



2. Short answers. . . Intuition and Understanding

(a) If you are driving, then what device (or devices) in your car will be the best way to change your tangential acceleration?

Solution: Gas pedal & brake

(b) What is the curvature of a circle with radius 64?

Solution: $\kappa = \frac{1}{64}$

(c) $f(x, y)$ is defined to equal 5 for all points on the disk $(x-1)^2 + (y-2)^2 \leq 16$, to equal -3 for all points on the disk $(x+5)^2+(y+8)^2 \leq 4$, and to equal 0 everywhere else. Compute:

$$\int_{x=-90}^{100} \int_{y=-100}^{90} f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

Solution: $= 5 \cdot \pi(4^2) - 3\pi(2^2) = \boxed{68\pi}$

(d) Will the surface integral

$$\iint_S f(x, y, z)\,\mathrm{d}S$$

typically give you the surface area of S? Explain your answer in one sentence or less.

Solution: No. Only if $f(x, y, z) = 1$ will the integral compute the surface area of S.

(e) What is the average value of the function $f(x, y) = 4+2y$ on the rectangle $2 \leq x \leq 6$, $1 \leq y \leq 3$?

Solution:

$$= \frac{\int_1^3 \int_2^6 (4 + 2x)\,\mathrm{d}x\,\mathrm{d}y}{(6-2)(3-1)} = \frac{\int_2^6 (4 + 2x)\,\mathrm{d}x}{4} = \frac{\left[4x + x^2\right]_2^6}{4} = \boxed{12}$$

3. Short answers... Definitions and Theorems

(a) Suppose that $\nabla f(0,0) = \langle 0, 0 \rangle$, and $f_{xx}(0,0)$ and $f_{yy}(0,0)$ are both negative. Do you need anything else to conclude that $(0,0)$ is a local maximum? (If yes, then what? If no, then why not?)

> **Solution:** Yes, need to know that the discriminant > 0. In particular, that $f_{xx}(0,0)f_{yy}(0,0) > f_{x,y}(0,0)^2$.

(b) What does it mean (definition!) for a vector field $\mathbf{F}(x, y, z)$ to be incompressible?

> **Solution:** $\operatorname{div} \mathbf{F} = 0$.

(c) According to the theorem that we learned, if f is a continuous function on a set Ω, then what condition or conditions on Ω will guarantee that f attains an absolute maximum and absolute minimum?

> **Solution:** Ω must be closed and bounded

(d) Assume that you have been given a differentiable vector field defined on the first octant. How can you quickly tell if it is conservative?

> **Solution:** Since it is defined on the entire first octant (a simply connected domain), the vector field is conservative if and only if its curl is the zero vector.

4. A certain differentiable function satisfies:

   (a) $f(2,-4) = 1$, and $f(-9,6) = 7$

   (b) $\nabla f(2,-4) = (5,3)$, and $\nabla f = (-9,6) = (8,-\pi)$.

   At each of the two points in question (i.e. at $(2,-4)$ and at $(-9,6)$) answer the following questions:

   (a) In what direction is the function increasing the fastest?

   > **Solution:**
   > For $(2,-4)$: $\langle 5, 3\rangle$;
   > For $(-9,6)$: $\langle 8, -\pi\rangle$.

   (b) What is the rate of change in that direction?

   > **Solution:**
   > For $(2,-4)$: $\sqrt{25+9} = \sqrt{34}$;
   > For $(-9,6)$: $\sqrt{64+\pi^2}$.

   (c) What is the directional derivative in the direction of $\langle 4,-3\rangle$? (Note: just to be completely clear about semantics here, you are supposed to give the same directional derivative at each point. I did not ask for the directional derivative in the direction of the point $(4,-3)$.)

   > **Solution:** Let $\mathbf{u} = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$.
   > For $(2,-4)$: $D_{\mathbf{u}}f(2,-4) = \nabla f(2,-4)\cdot\mathbf{u} = \langle 5,3\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = \frac{11}{5}$.
   > For $(-9,6)$: $D_{\mathbf{u}}f(-9,6) = \nabla f(-9,6)\cdot\mathbf{u} = \langle 8,-\pi\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = \frac{32+3\pi}{5}$.

   (d) What is the tangent plane and/or the linear approximation at each of the two points?

   > **Solution:**
   > For $(2,-4)$: $L(x,y) = 1 + 5(x-2) + 3(y+4)$
   > For $(-9,6)$: $L(x,y) = 7 + 8(x+9) - \pi(y-6)$

5. Find the maximum and minimum of the function

$$f(x,y) = x - 4x^2 + 3y - 4y^2$$

on the set

$$g(x,y) = x^2 + y^2 \le 40.$$

Show your work carefully, and explain what you are doing. (No essays, please. Just a few short words in the right places will suffice.)

Solution: First, consider $g < 40$. Setting $\nabla f = 0$, and solving

$$\nabla f = \langle 1 - 8x, 3 - 8y \rangle = \mathbf{0}$$

gives the critical point $(\frac{1}{8}, \frac{3}{8})$. Next, considering $g = 40$, we solve $\nabla f = \lambda \nabla g$:

$$\begin{cases} 1 - 8x = 2\lambda x \\ 3 - 8y = 2\lambda y \\ x^2 + y^2 = 40 \end{cases}$$

The first two equations give

$$x = \frac{1}{8+2\lambda} \qquad y = \frac{3}{8+2\lambda}$$

Substituting into the the third equation gives

$$\frac{10}{(8+2\lambda)^2} = 40 \implies \frac{1}{(8+2\lambda)^2} = 4 \implies \frac{1}{8+2\lambda} = \pm 2$$

which yields the two constrained critical points: $(2,6)$, $(-2,-6)$. Evaluating f on the found critical points:

$$f(\frac{1}{8}, \frac{3}{8}) = \frac{5}{8}$$
$$f(2,6) = -140$$
$$f(-2,-6) = -180$$

Thus the maximum is $\frac{5}{8}$, and the minimum is -180.

6. Let S be the part of the set

$$z = x^2 + y^2$$

which is between the planes $z = 9$ and $z = 16$.
Express the surface area for S as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ which has **constant** bounds of integration. (i.e. it should be over a rectangular solid or a rectangle in the domain in which you are finally integrating.) You do **NOT** need to find this integral.

Solution: The set S can be parametrized by

$$G(r, \theta) = (r\cos\theta, r\sin\theta, r^2) \qquad 3 \le r \le 4,\ 0 \le \theta \le 2\pi$$

We can then compute

$$\begin{aligned} \mathbf{N} = G_r \times G_\theta &= \langle \cos\theta, \sin\theta, 2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle \\ &= \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle \end{aligned}$$

and

$$\|\mathbf{N}\| = \sqrt{4r^4 + r^2}$$

The surface area of S can thus be computed as the surface integral

$$\iint_S 1\, \mathrm{d}S = \boxed{\int_0^{2\pi} \int_3^4 \sqrt{4r^4 + r^2}\, \mathrm{d}r\, \mathrm{d}\theta}$$

7. Let C be the curve given by

$$\mathbf{r}(t) = \left(t \cdot \cos(5\pi t), \frac{t^3}{4+t^2}, t + \sin(5\pi t) \right),$$

with $0 \le t \le 2$. Compute the following integral:

$$\int_C \left\langle y, x, 3z^2 \right\rangle \cdot \mathrm{d}\mathbf{r}.$$

Solution: Note that the vector field $\mathbf{F} = \left\langle y, x, 3z^2 \right\rangle$ is conservative, with potential function $f = xy + z^3$. Hence by the Fundamental Theorem for Line Integrals,

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= f(\mathbf{r}(2)) - f(\mathbf{r}(0)) \\ &= f(2,1,2) - f(0,0,0) \\ &= 2 + 8 - 0 = \boxed{10} \end{aligned}$$

8. Let Q be the set of points within the set:

$$\{(x, y, z) : x^2 + y^2 + z^2 \le 9, \quad \text{and } 0 \le y\}$$

and let ∂Q be the boundary of this set. If $\mathbf{n}$ is the outward unit normal to this region, then compute:

$$\iint_{\partial Q} (\cos(z^4), y^2, \sin(x^4)) \cdot \mathbf{n}\, \mathrm{d}S.$$

Solution: By the divergence theorem,

$$\iint_{\partial Q} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_Q \operatorname{div} \mathbf{F}\, \mathrm{d}V$$

so

$$\iint_{\partial Q} (\cos(z^4), y^2, \sin(x^4)) \cdot \mathbf{n}\, \mathrm{d}S = \iiint_Q 2y\, \mathrm{d}V$$

Evaluating this integral in spherical coordinates,

$$\begin{aligned}
\iiint_Q 2y\, \mathrm{d}V &= \int_0^{\pi} \int_0^{\pi} \int_0^3 2\rho \sin\varphi \sin\theta \cdot \rho^2 \sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\theta \\
&= \int_0^3 2\rho^3\, \mathrm{d}\rho \cdot \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \cdot \int_0^{\pi} \sin^2\varphi\, \mathrm{d}\varphi \\
&= \left[\frac{\rho^4}{2}\right]_0^3 \cdot [-\cos\theta]_0^{\pi} \cdot \left[\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right]_0^{\pi} \\
&= \frac{81}{2} \cdot (1+1) \cdot (\frac{\pi}{2} - 0 - 0) \\
&= \boxed{\frac{81\pi}{2}}
\end{aligned}$$

9. Let E be the subset of

$$z = x^2 + y^2$$

which also satisfies

$$z \leq 16, \qquad x \geq 0, \quad \text{and} \quad y \leq 0.$$

Express

$$\iint_E y^2 \, \mathrm{d}S$$

as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. You do **NOT** need to find this integral.

Solution: The set E can be parametrized by

$$G(r, \theta) = (r\cos\theta, r\sin\theta, r^2) \qquad 0 \leq r \leq 4, \quad -\frac{\pi}{2} \leq \theta \leq 0$$

Then we can compute

$$\begin{aligned} \mathbf{N} = G_r \times G_\theta &= \langle \cos\theta, \sin\theta, 2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle \\ &= \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle \end{aligned}$$

and

$$\|\mathbf{N}\| = \sqrt{4r^4 + r^2}$$

Which allows to rewrite the integral as

$$\iint_E y^2 \, \mathrm{d}S = \boxed{\int_{-\frac{\pi}{2}}^{0} \int_0^4 r^2 \sin^2\theta \cdot \sqrt{4r^4 + r^2} \, \mathrm{d}r \, \mathrm{d}\theta}$$

10. Let E be the part of the set

$$\sqrt{x^2+y^2} \leq z \leq 3$$

that also satisfies

$$x \leq 0.$$

Find

$$\iiint_E x \,\mathrm{d}V.$$

Solution: The inequaltiy can be reexpressed as $0 \leq r \leq z \leq 3$. The bounds for the variables are

$$\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \qquad r \in [0,3] \qquad z \in [r,3]$$

E is half of a solid cone. The integral can be written in cylindrical coordinates as

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^3 \int_r^3 r\cos\theta \cdot r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \,\mathrm{d}\theta \cdot \int_0^3 \int_r^3 r^2 \,\mathrm{d}z\,\mathrm{d}r \\
&= [\sin\theta]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cdot \int_0^3 r^2(3-r)\,\mathrm{d}r \\
&= (-1-1)\cdot \left[r^3 - \frac{r^4}{4}\right]_0^3 \\
&= -2\left(27 - \frac{81}{4}\right) = \boxed{-\frac{27}{2}}
\end{aligned}$$