1. For the following questions, suppose $\mathbf{u} = \langle 1, 1, 1 \rangle$ and $\mathbf{v} = \langle 0, -1, 0 \rangle$.

(a) (5 points) Evaluate $2\mathbf{u} + \mathbf{v}$.

Solution: $= \langle 2, 2, 2 \rangle + \langle 0, -1, 0 \rangle = \boxed{\langle 2, 1, 2 \rangle}$

(b) (5 points) Evaluate $\mathbf{u} \cdot \mathbf{v}$.

Solution: $= 1(0) + 1(-1) + 1(0) = \boxed{-1}$

(c) (5 points) Find $\cos(\theta)$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution: $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \dfrac{-1}{\sqrt{3} \cdot 1} = \boxed{-\dfrac{1}{\sqrt{3}}}$

(d) (5 points) Evaluate $\mathbf{u} \times \mathbf{v}$.

Solution: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \boxed{\langle 1, 0, -1 \rangle}$

(e) (5 points) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Solution: $= \|\mathbf{u} \times \mathbf{v}\| = \big\| \langle 1, 0, -1 \rangle \big\| = \boxed{\sqrt{2}}$

(f) (5 points) Find the distance between the point $(1, 0, 0)$ and the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ which passes through the origin.

Solution: Let $O = (0, 0, 0)$ and $S = (1, 0, 0)$.
Using as a normal vector for this plane $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 1, 0, -1 \rangle$, the distance can be computed as

$$\left\| \text{proj}_{\mathbf{n}} \overrightarrow{OS} \right\| = \frac{|\overrightarrow{OS} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle 1, 0, 0 \rangle \cdot \langle 1, 0, -1 \rangle|}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}}$$

2. Solve the Problems regarding the point $P = (1, 1, -1)$, $Q = (1, 1, 1)$ and the vector $\mathbf{v} = \langle 2, 1, 1 \rangle$.

   (a) (5 points) Find a parametric equation for the line ℓ passing through P with direction vector $\mathbf{v}$.

   Solution: $\mathbf{r}(t) = \langle 1, 1, 1 \rangle + t \langle 2, 1, 1 \rangle$

   (b) (5 points) Is Q on the line found in part (a)? Explain your response.

   Solution: No. There is no t for which $\mathbf{r}(t) = \langle 1, 1, 1 \rangle$.
   Alternatively, one can note that two points uniquely determine a line between them. A line passing through both P and Q must have direction vector that is a scalar multiple of $\langle 0, 0, 1 \rangle$.

   (c) (5 points) Find the equation for the plane passing through Q and with normal vector $\mathbf{v}$.

   Solution: Any of the following:

   $$\langle 2, 1, 1 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0$$
   $$2(x - 1) + y - 1 + z - 1 = 0$$
   $$2x + y + z - 4 = 0$$

   (d) (5 points) Give an equation for a line that passes through P and is parallel to the plane found in part (c).

   Solution: There are many possible lines satisfying this property. Since such a line is parallel to the plane, it must be orthogonal to the normal vector of the plane, $\mathbf{n} = \langle 2, 1, 1 \rangle$. Hence the direction vector $\mathbf{d}$ of the line must have the property that $\mathbf{d} \cdot \mathbf{n} = 0$.
   One possible answer: Choosing $\mathbf{d} = \langle 1, -1, -1 \rangle$ gives the line

   $$\mathbf{r}(t) = \langle 1, 1, -1 \rangle + t \langle 1, -1, -1 \rangle .$$

3. (a) (5 points) Sketch and describe the trace of the intersection of the plane $z = -3$ with the surface $x^2 + y^2 - z^2 = 16$.

Solution: The trace is given by $x^2 + y^2 = 7$. This is a circle of radius $\sqrt{7}$, centered at the origin.

(b) (5 points) Give the inequalities in spherical coordinates that describe the upper unit half ball, which in Cartesian coordinates is described by:

$$z \geq 0, \qquad x^2 + y^2 + z^2 \leq 1$$

Solution:

$$0 \leq \rho \leq 1 \qquad 0 \leq \varphi \leq \frac{\pi}{2} \qquad 0 \leq \theta \leq 2\pi$$

4. (10 points) Given the vectors $\mathbf{u} = \langle 2, 1, 0 \rangle$, $\mathbf{v} = \langle 0, -1, 1 \rangle$ and the parameterized line $\mathbf{r}(t) = \langle t, 1-t, 2t \rangle$, find all values of t for which the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ is equal to the volume of the paraallelopiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{r}(t)$ (ignoring units).

Solution: This can be rewritten as: Find all values of t such that

$$\|\mathbf{u} \times \mathbf{v}\| = \left|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{r}(t)\right|$$

Computing gives

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \langle 1, -2, -2 \rangle$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{1+4+4} = 3$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{r}(t) = t - 2(1-t) - 4t = -t - 2$$

So the equation to solve is

$$3 = |-t-2|$$

which has solutions $\boxed{t = -5, 1}$

5. Answer the following questions concerning the vector valued function

$$\mathbf{r}(t) = \langle \ln(t), t^2, e^{t-1} \rangle$$

for $t > 0$.

(a) (5 points) Evaluate

$$\lim_{t \to 1} \mathbf{r}(t).$$

Solution: $\boxed{\langle 0, 1, 1 \rangle}$

(b) (5 points) Evaluate $\mathbf{r}'(t)$.

Solution: $\boxed{\left\langle \frac{1}{t}, 2t, e^{t-1} \right\rangle}$

(c) (5 points) Evaluate

$$\int_1^2 \mathbf{r}(t)\,\mathrm{d}t.$$

Solution: Integrating the first component function requires integration by parts:

$$\int_1^2 \ln(t)\,\mathrm{d}t = t\ln t - t\Big|_1^2 = 2\ln 2 - 1$$

Integrating the other two component functions is straightforward, giving the answer:

$$\boxed{\left\langle 2\ln 2 - 1, \frac{7}{3}, e - 1 \right\rangle}$$

6. Consider the vector valued function

$$\mathbf{r}(t) = \langle \sin(4t), \cos(4t), 3t \rangle$$

for $0 \leq t \leq 4$.

(a) (5 points) Find the arc-length function $s(t)$ of $\mathbf{r}(t)$.

Solution:

$$\mathbf{r}' = \langle 4\cos(4t), -4\sin(4t), 3 \rangle$$
$$\|\mathbf{r}'\| = \sqrt{16\cos^2(4t) + 16\sin^2(4t) + 9} = \sqrt{16+9} = 5$$

so

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,\mathrm{d}u = \int_0^t 5\,\mathrm{d}u$$
$$\implies \boxed{s(t) = 5t}$$

(b) (5 points) What is the length of the curve parametrized by $\mathbf{r}(t)$?

Solution: $s(4) = \boxed{20}$

(c) (5 points) Find the arc-length parametrization $\mathbf{r}(s)$.

Solution: Since $s = 5t \implies t = \frac{s}{5}$, $\boxed{\mathbf{r}(s) = \left\langle \sin\left(\frac{4}{5}s\right), \cos\left(\frac{4}{5}s\right), \frac{3}{5}s \right\rangle}$