1. Find the limit, if it exists. If the limit does not exist, explain why.

   (a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$$

Solution: Approaching along the x-axis, the limit reduces to

$$\lim_{x\to 0} \frac{1}{x}$$

which does not exist. Hence the original limit $\boxed{\text{does not exist}}$

   (b) (5 points)

$$\lim_{(x,y)\to(0,0)} e^{-\frac{1}{x^2+y^2}}$$

Solution: Converting to polar, the limit becomes

$$\lim_{r\to 0} e^{-\frac{1}{r^2}} = \boxed{0}$$

   (c) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$$

Solution: Converting to polar, the limit becomes

$$\lim_{r\to 0} \frac{r^4\cos\theta\sin^3\theta}{r^2} = \lim_{r\to 0} r^2\cos\theta\sin^3\theta = \boxed{0}$$

since $|\cos\theta\sin^3\theta| \leq 1$.

2. Evaluate the partial derivatives, if they exist. If they do not exist, explain why.

(a) (5 points) $f_y(1,-2)$ for $f(x,y) = xy + \sin(\pi xy)$.

Solution:

$$f_y = x + \cos(\pi xy) \cdot \pi x$$
$$f_y(1,-2) = 1 + \pi\cos(-2\pi) = \boxed{\pi + 1}$$

(b) (5 points) $\frac{\partial^3 f}{\partial y \partial x^2}(0,0)$ for $f(x,y) = x^3 + x^2 + x + 1 + y^3 + y^2 + x^2 y$

Solution: The mixed derivative will make all terms go to zero, except for x^2y, which goes to $\boxed{2}$

(c) (5 points) $f_z(0,1,0)$ for

$$f(x,y,z) = 2y|x| + 3x|z| + 4z|y| + 5z.$$

Solution: Since the partial derivative w.r.t. z treats x and y as constants, we can fix the function at $x = 0, y = 1$ and then take the partial dervative w.r.t. z and then evaluate at $z = 0$:

$$f(0,1,z) = 4z + 5z = 9z$$
$$f_z(0,1,z) = 9$$
$$f_z(0,1,0) = \boxed{9}$$

3. Consider the function $f(x, y) = e^{xy} + 3x$.

(a) (5 points) Give the linearization $L(x, y)$ of $f(x, y)$ at $(2, 0)$.

Solution:

$$\begin{aligned} f(2,0) &= e^0 + 6 = 7 \\ f_x &= e^{xy}y + 3 \\ f_x(2,0) &= 3 \\ f_y &= e^{xy}x \\ f_y(2,0) &= 2e^0 = 2 \end{aligned}$$

so

$$\begin{aligned} L(x,y) &= f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0) \\ &\boxed{L(x,y) = 7 + 3(x-2) + 2y} \end{aligned}$$

(b) (5 points) Use your result from part (a) to write the equation of the tangent plane to the graph of $f(x, y)$ at $(2, 0)$.

Solution:

$$\boxed{z = 7 + 3(x-2) + 2y}$$

4. Let

$$f(x,y,z) = x^2 + 4y^2 - z^2$$

(a) (5 points) Find the gradient of f.

Solution:

$$\nabla f = \boxed{\langle 2x, 8y, -2z\rangle}$$

(b) (5 points) Which points on the level surface

$$f(x,y,z) = 1$$

have tangent planes parallel to the plane $y = 0$.

Solution: Since a normal vector for the plane $y = 0$ is $\langle 0, 1, 0\rangle$,

$$\nabla f = \lambda \langle 0, 1, 0\rangle \implies x, z = 0,$$

so the points on the level surface with the correct tangent planes occur when $x, z = 0$. To find all such points on the level surface, we do:

$$f(0,y,0) = 4y^2 = 1 \implies y = \pm\frac{1}{2}$$

hence $\boxed{(0, \pm\frac{1}{2}, 0)}$

(c) (5 points) Give an example of a vector $\mathbf{v}$ for which f is decreasing in the direction of $\mathbf{v}$ starting at $(3, 2, 1)$.

Solution:

$$-\nabla f(3,2,1) = -\langle 6, 16, -2\rangle = \boxed{\langle -6, -16, 2\rangle}$$

5. (15 points) Let
$$f(r,\theta,z) = r^2 \sin\theta$$
be written in cylindrical coordinates. Use the chain rule to calculate $\frac{\partial f}{\partial x}$ at the point P which is $(x,y,z) = (-1,0,1)$ in Cartesian coordinates. Confirm your result by expressing f in Cartesian coordinates and taking its partial derivatives with respect to x.

Solution: The coordinate in cylindrical is $(r,\theta,z) = (1,\pi,0)$. Using the conversion formulas
$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$
the partial derivatives $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial \theta}{\partial x}$ can be calculated as needed for the chain rule:
$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \cancel{\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}} \\
&= 2r\sin\theta \cdot \frac{x}{\sqrt{x^2+y^2}} + r^2\cos\theta \cdot \frac{-y}{x^2+y^2} \\
&= \cancel{2\sin\pi \cdot \frac{-1}{1}} + \cancel{\cos\pi \cdot \frac{0}{1}} \\
&= \boxed{0}
\end{aligned}$$

Alternatively, converting f to Cartesian coordinates,
$$f = r^2\sin\theta \rightsquigarrow (x^2+y^2)\frac{y}{\sqrt{x^2+y^2}} = y\sqrt{x^2+y^2}$$
so
$$f_x = \frac{xy}{\sqrt{x^2+y^2}} = \frac{0}{1} = \boxed{0}$$

6. Consider the function
$$f(x, y) = x^3 - 3x - y^3 + 3y$$
on the rectangular domain $\mathcal{D}$ defined by
$$-\frac{3}{2} \leq x \leq \frac{3}{2}, \qquad -\frac{3}{2} \leq y \leq 0$$

(a) (5 points) Find the critical points of f in the interior of $\mathcal{D}$.

Solution:

$$\nabla f = 0 \implies \begin{cases} 3x^2 - 3 = 3(x^2 - 1) = 0 \\ -3y^2 + 3 = -3(y^2 - 1) = 0 \end{cases} \implies \begin{matrix} x = \pm 1 \\ y = \pm 1 \end{matrix}$$

Restricted to $\mathcal{D}$, the critical points are $\boxed{(\pm 1, -1)}$

(b) (5 points) Describe the local behavior of f near the critical points.

Solution: Computing the discriminant:

$$\text{disc.} = f_{xx} f_{yy} - (f_{xy})^2 = (6x)(-6y) - 0 = -36xy$$

So

- For $(1, -1)$, the discriminant is $36 > 0$. Since $f_{xx}(1, -1) = 6 > 0$, $\boxed{(1, -1) \text{ is a local min}}$
- For $(-1, -1)$, the discriminant is $-36 < 0$, so $\boxed{(-1, -1) \text{ is a saddle point}}$

(c) (5 points) The global maximum and minimum values of f on the boundary of the rectangle are 2 and $-\frac{25}{8}$ respectively. Find the global maximum value and the global minimum value for f on $\mathcal{D}$ if they exist. Explain your response.

Solution: Since $f(1, -1) = -4$ is a local min in $\mathcal{D}$, the global minimum on $\mathcal{D}$ is $\min(-\frac{25}{8}, -4) = \boxed{-4}$. Since there is no local max in the interior of $\mathcal{D}$, the global maximum on $\mathcal{D}$ is found on the boundary, and is $\boxed{2}$

7. (15 points) Use Lagrange multipliers to find the critical points of the function

$$f(x, y, z) = 2x + 2y - z$$

on the sphere

$$x^2 + y^2 + z^2 = 4.$$

Identify the global maximum and minimum values of f on the sphere.

Solution: Letting $g(x, y, z) = x^2 + y^2 + z^2 - 4$,

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ \langle 2, 2, -1 \rangle &= \lambda \langle 2x, 2y, 2z \rangle \implies \begin{cases} 2 = 2\lambda x \\ 2 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \\ &\implies \begin{cases} x = 1/\lambda \\ y = 1/\lambda \\ z = -1/(2\lambda) \end{cases} \end{aligned}$$

then substituting into the constraint,

$$\begin{aligned} \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} &= 4 \\ \frac{1}{\lambda^2}(2 + \frac{1}{4}) &= 4 \\ \frac{1}{\lambda^2} &= \frac{16}{9} \\ \lambda &= \pm\frac{3}{4} \end{aligned}$$

For $\lambda = 3/4$, $f(4/3, 4/3, -2/3) = 6$. For $\lambda = -3/4$, $f(-4/3, -4/3, 2/3) = -6$. Thus on the sphere, $\boxed{\text{the global max is 6, and the global min is } -6}$