Math 222 Final Exam A December 18, 2019

Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. You will have 1 hour and 50 minutes to complete this exam.

Question	Points	Score
1	15	
2	10	
3	20	
4	15	
5	10	
6	20	
7	20	
8	15	
9	15	
Total:	140	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. Let $\mathbf{u} = \langle 1, 2, 1 \rangle$ and $\mathbf{v} = \langle 1, 1, -1 \rangle$.

   (a) (5 points) Find a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   (b) (5 points) Compute the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

   (c) (5 points) Give the equation for the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $(1, 1, 1)$.

2. Consider the curve $\mathcal{C}$ given by the parametrization

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \quad \text{for } 1 \leq t \leq 4$$

(a) (5 points) Find the speed of $\mathbf{r}(t)$ as a function of t.

(b) (5 points) Compute the scalar line integral

$$\int_{\mathcal{C}} \frac{1}{\sqrt{1 + 2x^2 + 2y + 4y^2 + 5xz}}\, ds.$$

3. Calculate the following quantities if they exist. Otherwise, explain why they do not exist. Justify either response.

(a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$$

(b) (5 points) For

$$f(x,y,z) = e^{xz} - \sin(y^2+z^2) + x\ln(1+z^4) - xyz$$

compute

$$f_{xyz}(x,y,z)$$

(c) (5 points) For $f(x, y, z) = 2x + y + z + xyz$ find the unit vector in the direction for which f increases the most starting at the point $(1, 1, 0)$.

(d) (5 points) Find the equation for the tangent plane to the surface

$$x^2 + y^2 - z^2 = 4$$

at the point $(-1, 2, 1)$.

4. Let $\mathcal{D}$ be the disk of radius 2. I.e. the set of points (x, y) with $x^2 + y^2 \leq 4$. Take

$$f(x, y) = x^2 + y^2 - xy.$$

(a) (5 points) Find the critical points of $f(x, y)$ in the interior of $\mathcal{D}$.

(b) (5 points) Describe the local behavior of $f(x, y)$ at the critical points found in part (a).

(c) (5 points) Find the maximum and minimum values of f on $\mathcal{D}$.

5. (10 points) Let $\mathcal{W} = [1,3] \times [1,2] \times [0,2]$. Evaluate the triple integral

$$\iiint_{\mathcal{W}} \pi^2 xy \sin(\pi xz)\ dV$$

6. Evaluate the following integrals.

(a) (10 points) Let $\mathcal{D}$ be the region $x^2 + y^2 \leq 9$, $0 \leq x$, $y \leq 0$. Evaluate

$$\iint_{\mathcal{D}} y \; \mathrm{d}A.$$

(b) (10 points) Let $\mathcal{D}$ be the region $0 \leq x \leq \pi$ and $0 \leq y \leq \sin x$.

$$\iint_{\mathcal{D}} 1 + \cos^4 x \ \mathrm{d}A.$$

7. Let

$$\mathbf{F} = \langle x, 2y, -3z \rangle .$$

(a) (5 points) If $\mathbf{F}$ is a conservative vector field, find a potential. Otherwise, explain why it is not conservative.

(b) (5 points) Let $\mathcal{C}$ be the oriented curve with parametrization

$$\mathbf{r}(t) = \left\langle t - \sin \pi t, (t+1)^3, e^{t(1-t)} \right\rangle$$

for $0 \leq t \leq 1$. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

(c) (5 points) Verify that the vector field

$$\mathbf{A} = \langle 3yz, 0, yx \rangle$$

is a vector potential for $\mathbf{F}$ (a vector field $\mathbf{A}$ that satisfies $\mathbf{F} = \text{curl}(\mathbf{A})$).

(d) (5 points) Let $\mathcal{S}$ be an oriented surface with boundary on the plane $y = 0$. What is the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

State any theorems used in the computation and explain your response.

8. Let $\mathcal{D}$ be the right triangle with vertices $(-1,1),(-1,-1)$ and $(1,-1)$. Write $\mathcal{C}_1$ for the boundary side oriented from $(-1,1)$ to $(-1,-1)$, $\mathcal{C}_2$ for the boundary side oriented from $(-1,-1)$ to $(1,-1)$ and $\mathcal{C}_3$ for the boundary side oriented from $(1,-1)$ to $(-1,1)$ Let

$$\mathbf{F} = \langle y+2, x \rangle .$$

(a) (5 points) Calculate the vector line integral

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \, d\mathbf{r}$$

(b) (5 points) Calculate the vector line integral

$$\int_{C_2} \mathbf{F} \cdot \, d\mathbf{r}$$

(c) (5 points) Using only Green's Theorem and the computations in parts (a) and (b), compute the vector line integral

$$\int_{C_3} \mathbf{F} \cdot \, d\mathbf{r}$$

9. Let $\mathcal{S}$ be the boundary of the box $\mathcal{B} = [0,2] \times [0,2] \times [0,2]$ oriented outward and $\mathbf{F} = \langle x, yz, z^2 \rangle$.

   (a) (5 points) Compute $\text{div}(\mathbf{F})$.

   (b) (5 points) Calculate

$$\iiint_{\mathcal{B}} \text{div}(\mathbf{F})\ dV$$

(c) (5 points) The surface $\mathcal{S}$ consists of six squares. Three lie on one of the coordinate planes and a quick computation shows the vector surface integral

$$\iint_{\tilde{\mathcal{S}}} \mathbf{F} \cdot d\mathbf{S} = 0$$

for each of these. The face in the plane $x = 2$ has vector surface integral equal to 8 and the face in the plane $z = 2$ has vector surface integral equal to 16. Let $\mathcal{S}'$ be the remaining face on the $y = 2$ plane. Using only these facts, the Divergence Theorem and your result from part (b), compute the vector surface integral

$$\iint_{\mathcal{S}'} \mathbf{F} \cdot \, d\mathbf{S}$$

Coordinate systems

Polar	Cylindrical	Spherical
$x = r\cos(\theta)$ $y = r\sin(\theta)$	$x = r\cos(\theta)$ $y = r\sin(\theta)$ $z = z$	$x = \rho\cos(\theta)\sin(\phi)$ $y = \rho\sin(\theta)\sin(\phi)$ $z = \rho\cos(\phi)$
$r = \sqrt{x^2+y^2}$ $\tan(\theta) = \frac{y}{x}$	$r = \sqrt{x^2+y^2}$ $\tan(\theta) = \frac{y}{x}$ $z = z$	$\rho = \sqrt{x^2+y^2+z^2}$ $\tan(\theta) = \frac{y}{x}$ $\cot(\phi) = \frac{z}{\sqrt{x^2+y^2}}$
$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

Unit vectors

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \mathbf{N} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Useful area and volume formulas

$$\text{Surface area of sphere of radius } R = 4\pi R^2$$

$$\text{Volume of sphere of radius } R = \frac{4}{3}\pi R^3$$

Derivative formulas

$$\text{Directional derivative}: D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot\mathbf{u},$$

$$\text{Discriminant}: D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$$

Trig identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$$

Change of variables

$$G : \mathcal{D}_0 \to \mathcal{D}$$

$$G(u,v) = (x(u,v), y(u,v))$$

$$\mathrm{Jac}(G) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{D}_0} f(x(u,v), y(u,v))\ |\mathrm{Jac}(G)|\ \mathrm{d}u\,\mathrm{d}v$$

Line integrals

$$\mathbf{r}(t) \text{ for } a \le t \le b \text{ parametrizing } \mathcal{C}$$

$$\int_{\mathcal{C}} f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t$$

$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,\mathrm{d}t$$

Surface integrals

$$G(u,v) = (x(u,v), y(u,v), z(u,v)) \text{ for } (u,v)\in\mathcal{D} \text{ parametrizing } \mathcal{S}$$

$$\mathbf{n}(u,v) = \mathbf{t}_u \times \mathbf{t}_v$$

$$\iint_{\mathcal{S}} f(x,y,z)\,\mathrm{d}S = \iint_{\mathcal{D}} f(G(u,v))\,\|\mathbf{n}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v$$

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(u,v))\cdot\mathbf{n}(u,v)\,\mathrm{d}u\,\mathrm{d}v$$