Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. You will have 1 hour to complete this exam.

Question	Points	Score
1	30	
2	30	
3	20	
4	20	
Total:	100	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. For the following questions, suppose $\mathbf{u} = \langle 0, -1, 1\rangle$ and $\mathbf{v} = \langle -1, 0, 1\rangle$.

   (a) (5 points) Evaluate $\mathbf{u} - 3\mathbf{v}$.

   Solution: $\mathbf{u} - 3\mathbf{v} = \langle 3, -1, -2\rangle$.

   (b) (5 points) Evaluate $\mathbf{u} \cdot \mathbf{v}$.

   Solution: $\mathbf{u} \cdot \mathbf{v} = 1$.

   (c) (5 points) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

   Solution: We have that $1 = \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) = \sqrt{2} \cdot \sqrt{2} \cos(\theta)$ implying that $\cos(\theta) = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$.

   (d) (5 points) Evaluate $\mathbf{u} \times \mathbf{v}$.

   Solution:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= -\mathbf{i} - \mathbf{j} - \mathbf{k} \\ &= \langle -1, -1, -1\rangle . \end{aligned}$$

   (e) (5 points) Find the volume of the parallelepiped with adjacent edges given by $\mathbf{u}$, $\mathbf{v}$ and the standard unit vector $\mathbf{j}$.

   Solution: The volume is $|\mathbf{j} \cdot (\mathbf{u} \times \mathbf{v})| = 1$.

   (f) (5 points) Find the projection of $\mathbf{v}$ in the direction of $\mathbf{u}$.

   Solution: Calculating, we have $proj_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}/\|\mathbf{u}\|^2\mathbf{u} = \frac{1}{2}\mathbf{u} = \left\langle 0, -\frac{1}{2}, \frac{1}{2}\right\rangle$.

2. Solve the problems regarding the points $P = (1, 1, 3)$, $Q = (1, 1, 1)$ and $R = (1, -1, 1)$.

   (a) (8 points) Find a parametric equation for the line ℓ passing through P and Q.

   Solution: The direction vector $\vec{PQ} = \langle 0, 0, 2\rangle$ yields the equation $\mathbf{r}(t) = \langle 1, 1, 3\rangle + t\langle 0, 0, 2\rangle$.

(b) (7 points) Find the distance between the line in part (a) and the point R.

Solution: The distance equals $|\|\vec{PQ} \times \vec{RQ}\| / \|\vec{PQ}\|| = \| \langle 0, 0, 2 \rangle \times \langle 0, 2, 0 \rangle \| / 2 = 2$.

(c) (8 points) Find the equation for the plane passing through points P, Q and R.

Solution: A normal vector to such a plane is the direction vector of $\vec{PQ} \times \vec{RQ} = (-2\mathbf{k}) \times (2\mathbf{j}) = 4\mathbf{i}$ which was found to be $\langle -2, 2, 2 \rangle$ or, after scaling, $\langle -1, 1, 1 \rangle$. Since $\mathbf{i} \cdot \langle 1, 1, 3 \rangle = 1$, the equation is

$$x = 1.$$

(d) (7 points) Find the equation for the plane which passes through P and Q, and is perpendicular to the plane found in part (c).

Solution: The normal vector to this plane is normal to $\vec{PQ} = -2\mathbf{k}$ and the normal vector $\mathbf{i}$ to the plane found in part (c). Such a vector is $\vec{PQ} \times \mathbf{i} = -2\mathbf{j}$ or simply $\mathbf{j}$. Taking the dot product with $\langle 1, 1, 3 \rangle$ gives

$$y = 1.$$

3. (a) (10 points) Sketch and describe the trace of the intersection of the plane $y = 3$ with the surface $z^2 - x^2 - y^2 = 16$.

> **Solution:** This is a hyperboloid and, substituting $y = 3$ into the equation gives the intersection as $z^2 - x^2 = 25$ in the $y = 3$ plane which is a hyperbola.

(b) (10 points) The intersections of the surface $ax^2 + by^2 + cz^2 = 1$ with the plane $x = 1$ and the plane $z = 1$ are circles of radius 1. Find the real numbers a, b, c.

> **Solution:** For $x = 1$, we have the equation $by^2 + cz^2 = (1 - a)$ so that $\frac{b}{1-a} = 1 = \frac{c}{1-a}$ or $b = c = 1 - a$. For $z = 1$ we obtain $ax^2 + by^2 = 1 - c$ implying $a = b = (1 - c)$. Thus $a = b = 1 - a$ implies $a = 1/2$ and the remaining equalities imply that $b = c = 1/2$.

4. (a) (10 points) Does the equation $z^2 = x^2 + y^2$ in Cartesian coordinates describe the same surface as $z = r$ in cylindrical coordinates? If yes, explain why. If no, give equations that do.

Solution: They do not describe the same surface. This is because when z is negative, there is no solution to the cylindrical equation while there is a solution to the Cartesian equation (a circle). To correct this, the equation $r = |z|$ in cylindrical coordinates will describe the surface.

(b) (10 points) Find an equation of the form $\phi = f(\rho, \theta)$ in spherical coordinates for the surface $z = r$ given in cylindrical coordinates.

Solution: Substituting gives $\cos^2(\phi) = \sin^2(\phi)$ or $\tan^2(\phi) = 1$ so that $\phi = \frac{\pi}{4}$.

Some formulas

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle$$
$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$
$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - v_2 u_3, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

$$\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u}$$

Coordinate systems

Cylindrical

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

$$r = \sqrt{x^2 + y^2}$$
$$\tan(\theta) = \frac{y}{x}$$
$$z = z$$

Spherical

$$x = \rho\cos(\theta)\sin(\phi)$$
$$y = \rho\sin(\theta)\sin(\phi)$$
$$z = \rho\cos(\phi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan(\theta) = \frac{y}{x}$$
$$\cos(\phi) = \frac{z}{\rho}$$