Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. You will have 1 hour to complete this exam.

Question	Points	Score
1	30	
2	30	
3	20	
4	20	
Total:	100	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. For the following questions, suppose $\mathbf{u} = \langle 2, -1, 2 \rangle$ and $\mathbf{v} = \langle 1, 2, -2 \rangle$.

   (a) (5 points) Evaluate $2\mathbf{u} + \mathbf{v}$.

   Solution: $2\mathbf{u} + \mathbf{v} = \langle 5, 0, 2 \rangle$.

   (b) (5 points) Evaluate $\mathbf{u} \cdot \mathbf{v}$.

   Solution: $\mathbf{u} \cdot \mathbf{v} = -4$.

   (c) (5 points) Do the vectors $\mathbf{u}$ and $\mathbf{v}$ make an acute, right or obtuse angle? Justify your response.

   Solution: We have that $-4 = \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) = 9\cos(\theta)$ implying that $\cos(\theta) = -\frac{4}{9}$ so that they make an obtuse angle.

   (d) (5 points) Evaluate $\mathbf{u} \times \mathbf{v}$.

   Solution:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 1 & 2 & -2 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\
&= -2\mathbf{i} + 6\mathbf{j} + 5\mathbf{k} \\
&= \langle -2, 6, 5 \rangle .
\end{aligned}$$

   (e) (5 points) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

   Solution: The volume is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{4 + 36 + 25} = \sqrt{65}$.

   (f) (5 points) Find the projection of $\mathbf{v}$ in the direction of $\mathbf{u}$.

   Solution: Calculating, we have $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} / \|\mathbf{u}\|^2 \mathbf{u} = -\frac{4}{9}\mathbf{u} = \left\langle -\frac{8}{9}, \frac{4}{9}, -\frac{8}{9} \right\rangle$.

2. Solve the problems regarding the points $P = (2, 2, 0)$, $Q = (3, 0, 1)$ and $R = (2, -1, 1)$.

   (a) (8 points) Find a parametric equation for the line ℓ which is parallel to the line passing through P and Q and contains R.

   Solution: The direction vector $\vec{PQ} = \langle 1, -2, 1\rangle$ of the line passing through P and Q is also a direction vector of ℓ. This gives $\mathbf{r}(t) = \langle 2, -1, 1\rangle + t\,\langle 1, -2, 1\rangle$.

   (b) (7 points) Find the distance between the line in part (a) and the origin.

   Solution: Let $\mathbf{u}$ be the vector from the origin to R and $\mathbf{v}$ the direction vector obtained above. Then $\mathbf{v} \times \mathbf{u} = \langle -1, 1, 3\rangle$ which has norm $\sqrt{11}$. The distance equals $|\|\vec{v} \times \vec{u}\|/\|\vec{v}\|| = \sqrt{11}/\sqrt{6} = \frac{\sqrt{66}}{6}$.

(c) (8 points) Find the equation for the plane passing through points P, Q and R.

Solution: A normal vector to such a plane is the vector $\vec{PQ} \times \vec{RQ} = \langle 1, -2, 1 \rangle \times \langle 1, 1, 0 \rangle = \langle -1, 1, 3 \rangle$. Since $\langle -1, 1, 3 \rangle \cdot \langle 2, 2, 0 \rangle = 0$, the equation is

$$-x + y + 3z = 0 \qquad \text{or} \qquad x - y - 3z = 0.$$

(d) (7 points) Find the distance between the plane found in part (c) and the point $(0, 0, 1)$.

Solution: This is simply the norm of the projection of $\mathbf{k}$ to the normal vector which is

$$\frac{|\mathbf{k} \cdot \langle -1, 1, 3 \rangle|}{\| \langle -1, 1, 3 \rangle \|} = \frac{3}{\sqrt{11}} = \frac{3\sqrt{11}}{11}$$

3. (a) (10 points) Sketch and describe the trace of the intersection of the plane $y = -7$ with the surface

$$x^2 - y + z^2 = 16.$$

Solution: Substituting $y = -7$ into the equation gives the intersection as $z^2 + x^2 = 9$ which is a radius 3 circle centered about the origin.

(b) (10 points) For which values of t are the $x = t$ traces of the equation $x^2 + 2x + y^2 + z^2 = 1$ empty? Describe the graph of this equation.

Solution: For $x = t$, we have the equation $t^2 + 2t - 1 + y^2 + z^2 = 0$ or $y^2 + z^2 = -(t^2 + 2t - 1)$. So for all values $t^2 + 2t - 1 > 0$, the trace is empty. This inequality is solved by using the quadratic equation to find the roots which are $\frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$. Thus, whenever $t < -1 - \sqrt{2}$ or $t > -1 + \sqrt{2}$, the $x = t$ trace is empty.

This quadric surface is a sphere, as can be seen by completing the square. In particular, adding 1 to both sides of the equation gives

$$(x + 1)^2 + y^2 + z^2 = 2.$$

So this is a sphere of radius $\sqrt{2}$ centered at $(-1, 0, 0)$. This also gives an alternative way of determining when the $x = t$ trace is empty.

4. (a) (10 points) Convert the equation $y = zx$ to cylindrical coordinates and use this to show that if a point $P = (a, b, c)$ is on the graph (in Cartesian coordinates), then so is (ta, tb, c) for all positive t.

Solution: This equation converts to $z = \tan\theta$ after substitution. Since the equation is independent of r, rescaling in the radial direction will not affect whether you have a solution.

(b) (10 points) Find an equation in Cartesian coordinates for the equation

$$2\cos\varphi = \rho$$

and describe its graph.

Solution: Multiplying both sides by ρ gives $2\rho\cos\varphi = \rho^2$ which converts to $0 = x^2 + y^2 + z^2 - 2z = x^2 + y^2 + (z-1)^2 - 1$ or

$$x^2 + y^2 + (z-1)^2 = 1$$

which is the sphere of radius 1 centered at $(0, 0, 1)$.

Some formulas

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle$$
$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$
$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - v_2 u_3, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

$$\mathrm{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u}$$

Coordinate systems

Cylindrical	Spherical
$x = r\cos(\theta)$	$x = \rho\cos(\theta)\sin(\varphi)$
$y = r\sin(\theta)$	$y = \rho\sin(\theta)\sin(\varphi)$
$z = z$	$z = \rho\cos(\varphi)$
$r = \sqrt{x^2 + y^2}$	$\rho = \sqrt{x^2 + y^2 + z^2}$
$\tan(\theta) = \frac{y}{x}$	$\tan(\theta) = \frac{y}{x}$
$z = z$	$\cos(\varphi) = \frac{z}{\rho}$