Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. Be sure to show your work in every problem and explain your answer. You will have 1 hour to complete this exam.

Question	Points	Score
1	15	
2	20	
3	15	
4	20	
5	10	
6	20	
Total:	100	

Name: ________________________________

Recitation Instructor: ________________________________

Recitation Time: ________________________________

1. Answer the following questions concerning the vector valued function

$$\mathbf{r}(t) = \left\langle \frac{1}{t}, \cos(\pi t), \sin(\pi t) \right\rangle$$

for $t \neq 0$.

(a) (5 points) Evaluate

$$\lim_{t \to 1} \mathbf{r}(t).$$

Solution: The limit of a vector valued function can be found by taking the limit of its components.

$$\begin{aligned} \lim_{t \to 1} \mathbf{r}(t) &= \lim_{t \to 1} \left\langle \frac{1}{t}, \cos(\pi t), \sin(\pi t) \right\rangle, \\ &= \left\langle \lim_{t \to 1} \frac{1}{t}, \lim_{t \to 1} \cos(\pi t), \lim_{t \to 1} \sin(\pi t) \right\rangle, \\ &= \langle 1, -1, 0 \rangle. \end{aligned}$$

(b) (5 points) Evaluate $\mathbf{r}''(t)$.

Solution: The derivative of a vector valued function can be found by taking the limit of its components.

$$\begin{aligned} \mathbf{r}''(t) &= \left\langle \frac{d^2}{dt^2} \frac{1}{t}, \frac{d^2}{dt^2} \cos(\pi t), \frac{d^2}{dt^2} \sin(\pi t) \right\rangle, \\ &= \left\langle \frac{2}{t^3}, -\pi^2 \cos(\pi t), -\pi^2 \sin(\pi t) \right\rangle. \end{aligned}$$

(c) (5 points) Evaluate

$$\int_1^2 \mathbf{r}(t)\, \mathrm{d}t.$$

Solution: The integral of a vector valued function can be found by taking the limit of its components.

$$\begin{aligned}\int_1^2 \mathbf{r}(t)\,\mathrm{d}t &= \left\langle \int_1^2 \frac{1}{t}\,\mathrm{d}t, \int_1^2 \cos(\pi t)\,\mathrm{d}t, \int_1^2 \sin(\pi t)\,\mathrm{d}t \right\rangle, \\ &= \left\langle \ln(t)\big|_1^2, \left.\frac{\sin(\pi t)}{\pi}\right|_1^2, -\left.\frac{\cos(\pi t)}{\pi}\right|_1^2 \right\rangle, \\ &= \left\langle \ln(2), 0, -\frac{2}{\pi} \right\rangle.\end{aligned}$$

2. Consider the vector valued function

$$\mathbf{r}(t) = \left\langle t^{3/2}, \frac{3}{2}t, (t-1)^{3/2} \right\rangle$$

for $1 \leq t \leq 2$,

(a) (8 points) Find the arc-length function $s(t)$ of $\mathbf{r}(t)$.

Solution: Calculating the derivative gives

$$\mathbf{r}'(t) = \left\langle \frac{3}{2}\sqrt{t}, \frac{3}{2}, \frac{3}{2}\sqrt{t-1} \right\rangle$$

so that $\|\mathbf{r}'(t)\| = \frac{3}{2}\sqrt{t+1+t-1} = \frac{3}{2}\sqrt{2t}$ for all t. Thus

$$s(t) = \int_1^t \|\mathbf{r}'(u)\| du = \frac{3}{2}\int_1^t \sqrt{2u} du = \sqrt{2}(t^{3/2}-1)$$

(b) (7 points) What is the length of the curve parametrized by $\mathbf{r}(t)$?

Solution: The length is $s(4) = \sqrt{2}(4^{3/2}-1) = 7\sqrt{2}$.

(c) (5 points) Find the arc-length parametrization $\mathbf{r}(s)$.

Solution: Since $s = \sqrt{2}(t^{3/2}-1)$ we have $t = \left(\frac{s}{\sqrt{2}}+1\right)^{2/3}$. Substituting we have

$$\mathbf{r}(s) = \left\langle \frac{s}{\sqrt{2}}+1, \frac{3}{2}\left(\frac{s}{\sqrt{2}}+1\right)^{2/3}, \left(\left(\frac{s}{\sqrt{2}}+1\right)^{2/3}-1\right)^{3/2} \right\rangle. \tag{1}$$

3. Find the limit, if it exists. If the limit does not exist, explain why.

   (a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{|x||y|}{x^2+y^2}$$

> **Solution:** This limit does not exist. Checking along the y-axis gives zero, while along the line $x = y$ gives $1/2$.

   (b) (5 points)

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3}{x^2+y^2}\right)$$

> **Solution:** The limit inside of the cosine is 0 as one can check using polar coordinates. As cosine is a continuous function, this implies that the limit is $\cos(0) = 1$.

   (c) (5 points)

$$\lim_{(x,y,z)\to(0,0,1)} \frac{3x+2y+z}{x^2+2y^2+3z^2}$$

> **Solution:** This function is continuous at $(0, 0, 1)$ so that its value is $1/3$.

4. Evaluate the partial derivatives, if they exist. If they do not exist, explain why.

   (a) (6 points) $f_y(1, 1)$ for $f(x, y) = \sin(\pi xy) + x^2 + y^2$.

> **Solution:** The partial is $f_y(x, y) = \pi x \cos(\pi xy) + 2y$. So evaluating at $(1, 1)$ gives $f_y(1, 1) = 2 - \pi$.

   (b) (7 points) $f_{xy}(1, 1)$ for $f(x, y) = x \ln(y) + x^2 + y^2$.

> **Solution:** Calculating gives $f_x(x, y) = \ln(y) + 2x$ and then $f_{xy}(x, y) = \frac{1}{y}$. Evaluating at $(1, 1)$ gives 1

   (c) (7 points) $f_y(0, 0)$ for $f(x, y) = x + 2y|x| - 3y - 4x|y|$

> **Solution:** Along the y-axis, x is zero so that $f(0, y) = -3y$ and the partial derivative with respect to y is -3.

5. Consider the function $f(x, y) = xy + 2$.

(a) (5 points) Give the linearization $L(x, y)$ of $f(x, y)$ at $(1, -1)$.

Solution: The linearization is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

In this case, we have

$$L(x, y) = 1 - (x - 1) + (y + 1) = 3 - x + y.$$

(b) (5 points) Use your result from part (a) to write the equation for the tangent plane to the graph of $f(x, y)$ at $(1, -1)$.

Solution: The equation for the tangent plane is $z = L(x, y)$ or $x - y + z = 3$.

6. Let

$$f(x,y,z) = xe^y + ye^z + z$$

(a) (6 points) Find the gradient of f.

Solution: The gradient is $\nabla f(x,y,z) = \langle e^y, xe^y + e^z, ye^z + 1\rangle$.

(b) (7 points) Give the equation for the tangent plane of the level surface

$$f(x,y,z) = 1$$

at the point $(1,0,0)$.

Solution: Evaluating the gradient at this point gives $\nabla f(1,0,0) = \langle 1,2,1\rangle$. This is the normal to the tangent plane, which of course passes through $(1,0,0)$, yielding the equation

$$x + 2y + z = 1.$$

(c) (7 points) For what unit vector $\mathbf{u}$ is f is decreasing fastest when moving in the direction of $\mathbf{u}$ starting at $(1,0,0)$?

Solution:

The negative of the gradient vector $-\nabla f = \langle -1,-2,-1\rangle$ describes the direction of steepest descent. To make this a unit vector, we must normalize to obtain

$$\mathbf{u} = -\frac{\nabla f}{\|\nabla f\|} = \left\langle -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6} \right\rangle$$

Coordinate systems

Cylindrical

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

$$r = \sqrt{x^2 + y^2}$$
$$\tan(\theta) = \frac{y}{x}$$
$$z = z$$

Spherical

$$x = \rho\cos(\theta)\sin(\varphi)$$
$$y = \rho\sin(\theta)\sin(\varphi)$$
$$z = \rho\cos(\varphi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan(\theta) = \frac{y}{x}$$
$$\cos(\varphi) = \frac{z}{\rho}$$

Derivative formulas

Directional derivative : $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$