Math 222 **Practice Final Exam** **May 15, 2019**

Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. You will have 1 hour and 50 minutes to complete this exam.

Question	Points	Score
1	15	
2	10	
3	20	
4	15	
5	10	
6	20	
7	20	
8	15	
9	10	
Total:	135	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. Let $\mathbf{u} = \langle -1, 2, 0\rangle$, $\mathbf{v} = \langle 2, 0, 2\rangle$ and $\mathbf{w} = \langle 0, 3, 1\rangle$

   (a) (5 points) Compute the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

   Solution: We have $Area = \|\mathbf{u} \times \mathbf{v}\| = \| \langle 4, 2, -4\rangle \| = \sqrt{36} = 6$.

   (b) (5 points) Compute $\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

   Solution:
   $$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|v\|\|w\|} = \frac{2}{\sqrt{8}\sqrt{10}} = \frac{\sqrt{5}}{10}$$

   (c) (5 points) Give a parametric equation for the line perpendicular to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $(-1, 2, 3)$.

   Solution: Using the solution to part (a), we have that the line has direction vector $\langle 4, 2, -4\rangle$ or $\langle 2, 1, -2\rangle$. So a vector parameterization is
   $$\mathbf{r}(t) = \langle -1, 2, 3\rangle + t \langle 2, 1, -2\rangle .$$

2. Consider the curve $\mathcal{C}$ given by the parametrization

$$\mathbf{r}(t) = \left\langle 2t, t^2, t \right\rangle \qquad \text{for } 0 \le t \le 2$$

(a) (5 points) Find the unit tangent vector $\mathbf{T}(t)$ of $\mathbf{r}(t)$.

Solution: Observe $\mathbf{r}'(t) = \langle 2, 2t, 1\rangle$. so that $\|\mathbf{r}'(t)\| = \sqrt{4t^2+5}$ and

$$\mathbf{T}(t) = \frac{1}{\sqrt{4t^2+5}} \langle 2, 2t, 1\rangle$$

(b) (5 points) Compute the scalar line integral

$$\int_{\mathcal{C}} \sqrt{2y + xz + 5}\, \mathrm{d}s.$$

Solution: First observe that $\|\mathbf{r}'(t)\| = \sqrt{4t^2+5}$. So we have

$$\begin{aligned} \int_{\mathcal{C}} \sqrt{2y+xz+5}\mathrm{d}s &= \int_0^2 \sqrt{4t^2+5}\|\mathbf{r}'(t)\|\, \mathrm{d}\mathbf{t}, \\ &= \int_0^2 4t^2 + 5\, \mathrm{d}t, \\ &= \frac{4}{3}t^3 + 5t\Big|_0^2 = \frac{62}{3}. \end{aligned}$$

3. Calculate the following quantities if they exist. Otherwise, explain why they do not exist. Justify either response.

   (a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$$

Solution: This limit does not exist. Along the x-axis we have the limit is 1 but along the y-axis, the limit is 2.

   (b) (5 points) For

$$f(x,y,z) = e^{\sqrt{x^2+y^2}} + e^{\sqrt{z^2+y^2}} + 2xz$$

compute

$$f_{xz}(x,y,z)$$

Solution: The first term has z partial equal to zero and the second has x partial equal to zero, so the answer is 2.

(c) (5 points) For $f(x,y,z) = y\sin(x) - 4e^z$ find a unit vector pointing in the direction where f increases the fastest, starting at $(0,3,0)$.

Solution: The gradient of f is $\nabla f = \langle y\cos(x), \sin(x), -4e^z\rangle$ which, at $(0,3,0)$ evaluates to $\langle 3, 0, -4\rangle$. This points in the direction of greatest increase. To find the unit vector in that direction, we divide by the norm $= \sqrt{9+16} = 5$ to obtain

$$\left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle$$

(d) (5 points) Find the equation for the tangent plane to the surface

$$z = x^2 - y^2$$

at the point $(2,-1,3)$.

Solution: This is a level surface for $0 = x^2 - y^2 - z = f(x,y,z)$ and the gradient of f is $\nabla f = \langle 2x, -2y, -1\rangle$ which evaluates to $\langle 4, 2, -1\rangle$ at $(2,-1,3)$. As this is a normal vector to the tangent plane, we have that the equation is $4x + 2y - z = 4 \cdot 2 + 2 \cdot (-1) + (-1) \cdot 3 = 3$.

4. Let
$$f(x, y) = x^2 + \cos y$$
and
$$\mathcal{D} = [-1, 1] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(a) (5 points) Find the critical points of $f(x, y)$ in the interior of $\mathcal{D}$.

Solution: Solving $\nabla f = \langle 2x, -\sin y\rangle = \langle 0, 0\rangle$ gives $x = 0$ and $y = 0$ as the only critical point of f.

(b) (5 points) Describe the local behavior of $f(x, y)$ at the critical points found in part (a).

Solution: The Hessian of f has $2, -1$ along the diagonal and zeros off the diagonal. Thus $(0, 0)$ is a saddle point.

(c) (5 points) Find the global maximum and minimum values of $f(x, y) = x^2 + \cos y$ on $\mathcal{D}$.

Solution: Examining the horizontal sides of $\mathcal{D}$ where $y = \pm\pi/2$, we obtain $\cos(y) = 0$ and $f(x, y)$ is the function x^2 on $[-1, 1]$. This has a minimum value of 0 and maximum of 1. On the vertical sides we have $f(x, y) = \cos(y) + 1$ on $[-\pi/2, \pi/2]$ which has a minimum of 1 and maximum of 2. Thus the global min (max) are 0 (2) respectively.

5. (10 points) Let $\mathcal{W} = [0, \pi] \times [-1, 1] \times [2, 3]$. Evaluate the triple integral

$$\iiint_{\mathcal{W}} (3y^2 + z) \sin x \, \mathrm{d}V$$

Solution: This is a straightforward computation using Fubini and product of integrals

$$\begin{aligned}
\iiint_{\mathcal{W}} (3y^2 + z) \sin x \, \mathrm{d}V &= \left(\int_0^{\pi} \sin x \, \mathrm{d}x \right) \left(\int_{-1}^{1} \int_2^3 3y^2 + z \, \mathrm{d}z \, \mathrm{d}y \right), \\
&= 2 \int_{-1}^{1} 3zy^2 + \frac{1}{2} z^2 \Big|_2^3 \, \mathrm{d}y, \\
&= 2 \int_{-1}^{1} 3y^2 + \frac{5}{2} \, \mathrm{d}y, \\
&= 2 \left(y^3 + \frac{5}{2} y \right) \Big|_{-1}^{1}, \\
&= 14.
\end{aligned}$$

6. Evaluate the following integrals.

   (a) (10 points) Let $\mathcal{D}$ be the region $0 \leq x \leq 1 - y^2$. Evaluate

$$\iint_{\mathcal{D}} 1 + y^2 \ \mathrm{d}A.$$

Solution:

$$\begin{aligned}
\iint_{\mathcal{D}} 1 + y^2 \ \mathrm{d}A &= \int_{-1}^{1} \int_{0}^{1-y^2} 1 + y^2 \ \mathrm{d}x \ \mathrm{d}y, \\
&= \int_{-1}^{1} (1+y^2)(1-y^2) \ \mathrm{d}y, \\
&= \int_{-1}^{1} 1 - y^4 \ \mathrm{d}y, \\
&= y - \frac{y^5}{5} \bigg|_{-1}^{1} = 2 - \frac{2}{5} = \frac{8}{5}.
\end{aligned}$$

(b) (10 points) Let $\mathcal{E}$ be the half ball $x^2 + y^2 + z^2 \leq 1$ with $0 \leq z$ and evaluate

$$\iiint_{\mathcal{W}} z^2 \ \mathrm{d}V.$$

Solution:

$$\begin{aligned}
\iiint_{\mathcal{W}} z \ \mathrm{d}V &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \cos^2(\phi) \rho^2 \sin(\phi) \ \mathrm{d}\rho \ \mathrm{d}\phi \ \mathrm{d}\theta, \\
&= 2\pi \left(\int_0^{\pi/2} \cos^2(\phi) \sin(\phi) \ \mathrm{d}\phi \right) \left(\int_0^1 \rho^4 \ \mathrm{d}\rho \right), \\
&= \frac{2\pi}{5} \left(- \left. \frac{\cos^3(\phi)}{3} \right|_0^{\pi/2} \right), \\
&= \frac{2\pi}{15}.
\end{aligned}$$

7. Let

$$\mathbf{F} = \langle 2x, 2y, -4z \rangle .$$

(a) (5 points) If $\mathbf{F}$ is a conservative vector field, find a potential. Otherwise, explain why it is not conservative.

Solution: Yes. Computing curl($\mathbf{F}$) gives zero. One should derive and/or verify that $f(x, y, z) = x^2 + y^2 - 2z^2$ is a potential for $\mathbf{F}$.

(b) (5 points) Let $\mathcal{C}$ be an oriented curve from the origin to $(1, 2, 3)$. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Solution:

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= f(Q) - f(P), \\ &= f(1, 2, 3) - f(0, 0, 0) = -13. \end{aligned}$$

(c) (5 points) Verify that the vector field $\mathbf{A} = \langle 3zy, -zx, yx \rangle$ is a vector potential for $\mathbf{F}$, i.e. it satisfies $\mathbf{F} = \text{curl}(\mathbf{A})$.

Solution: Computing the divergence of $\mathbf{F}$ gives $\text{div}(\mathbf{F}) = 2+2-4 = 0$ so there is a vector potential (since $\mathbf{F}$ is defined on three space). One can check that

$$\text{curl}(\mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3zy & -zx & yx \end{vmatrix} = \mathbf{F}$$

(d) (5 points) Let $\mathcal{S}$ be the ellipsoid $x^2 + 4y^2 + 9z^2 = 1$ oriented outwardly. Compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

State any theorems used in the computation.

Solution: You could use Stokes' Theorem or the Divergence Theorem. By the fact that $\mathbf{F} = \text{curl}(\mathbf{A})$ we have $\text{div}(\mathbf{F}) = 0$. So if E is the interior of the ellipsoid, the divergence theorem gives

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{E} \text{div}(\mathbf{F})\ \mathrm{d}V = 0$$

8. Consider the annulus $\mathcal{D} = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$, the unit circle $\mathcal{C}_1$ oriented counter-clockwise, the radius 2 circle $\mathcal{C}_2$ oriented counter-clockwise (both circles centered at the origin) and the vector field

$$\mathbf{F} = \langle -y, x \rangle .$$

(a) (5 points) Compute the line integral

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$$

Solution: Parameterize by using $\mathbf{r}(t) = (\cos(t), \sin(t))$ gives

$$\begin{aligned}\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \ \mathrm{d}t, \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) \ \mathrm{d}t, \\ &= \int_0^{2\pi} \ \mathrm{d}t = 2\pi\end{aligned}$$

(b) (5 points) Using polar coordinates, find the area of $\mathcal{D}$ by computing the double integral

$$\iint_{\mathcal{D}} \mathrm{d}A$$

Solution:

$$\begin{aligned}\iint_A \mathrm{d}A &= \int_0^{2\pi} \int_1^2 r \ \mathrm{d}r \ \mathrm{d}\theta, \\ &= 2\pi \left. \frac{r^2}{2} \right|_1^2, \\ &= 3\pi.\end{aligned}$$

(c) (5 points) Using only Green's Theorem and the computations in parts (a) and (b), compute the vector line integral

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \ \mathrm{d}\mathbf{r}$$

Solution: Observing $\partial_x(x) - \partial_y(-y) = 2$, Green's Theorem says

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} - \int_{\mathcal{C}_1} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \iint_A 2 \, \mathrm{d}A,$$

or

$$\begin{aligned} \int_{\mathcal{C}_2} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} &= \int_{\mathcal{C}_1} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} + 2 \iint_A \, \mathrm{d}A, \\ &= 2\pi + 6\pi = 8\pi. \end{aligned}$$

9. Consider the vector field

$$\mathbf{F} = \left\langle e^{x^2+y^2}, y, z^2 - 2xze^{x^2+y^2} \right\rangle$$

and let $\mathcal{E}$ be the box $[0,3] \times [0,2] \times [0,1]$ and $\mathcal{S}$ its boundary oriented outwardly from $\mathcal{E}$.

(a) (5 points) Let $\mathcal{S}_1 = \{(x,y,1) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$ be the rectangular part of $\mathcal{S}$ where $z = 1$. Write the vector surface integral

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}$$

as an iterated integral. Do **not** evaluate the iterated integral.

Solution: At $z = 1$ the parametrization of $\mathcal{S}_1$ is simply $(x, y, 1)$ and the outward pointing unit normal is $\mathbf{k}$. So

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \int_0^3 \int_0^2 (1 - 2xe^{x^2+y^2}) \, \mathrm{d}y \, \mathrm{d}x.$$

(b) (5 points) Using the Divergence Theorem, compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}$$

Solution: Observe $\mathrm{div}(\mathbf{F}) = 2xe^{x^2+y^2} + 1 + 2z - 2xe^{x^2+y^2} = 2z + 1$. So

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} &= \iiint_{\mathcal{S}} \mathrm{div}(\mathbf{F}) \, \mathrm{d}V, \\
&= \int_0^1 \int_0^2 \int_0^3 2z + 1 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z, \\
&= 6 \left(z^2 + z \right) \Big|_0^1, \\
&= 12.
\end{aligned}$$

Coordinate systems

Polar	Cylindrical	Spherical
$x = r\cos(\theta)$	$x = r\cos(\theta)$	$x = \rho\cos(\theta)\sin(\phi)$
$y = r\sin(\theta)$	$y = r\sin(\theta)$	$y = \rho\sin(\theta)\sin(\phi)$
	$z = z$	$z = \rho\cos(\phi)$
$r = \sqrt{x^2+y^2}$	$r = \sqrt{x^2+y^2}$	$\rho = \sqrt{x^2+y^2+z^2}$
$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$
	$z = z$	$\cot(\phi) = \dfrac{z}{\sqrt{x^2+y^2}}$
$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

Unit vectors

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \mathbf{N} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Useful area and volume formulas

$$\text{Surface area of sphere of radius } R = 4\pi R^2$$
$$\text{Volume of sphere of radius } R = \frac{4}{3}\pi R^3$$

Derivative formulas

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

Trig identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$$

Change of variables

$$G : \mathcal{D}_0 \to \mathcal{D}$$

$$G(u,v) = (x(u,v), y(u,v))$$

$$\mathrm{Jac}(G) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{D}_0} f(x(u,v), y(u,v))\ |\mathrm{Jac}(G)|\ \mathrm{d}u\,\mathrm{d}v$$

Line integrals

$$\mathbf{r}(t) \text{ for } a \le t \le b \text{ parametrizing } \mathcal{C}$$

$$\int_{\mathcal{C}} f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t$$

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t$$

Surface integrals

$$G(u,v) = (x(u,v), y(u,v), z(u,v)) \text{ for } (u,v) \in \mathcal{D} \text{ parametrizing } \mathcal{S}$$

$$\mathbf{n}(u,v) = \mathbf{T}_u \times \mathbf{T}_v$$

$$\iint_{\mathcal{S}} f(x,y,z)\,\mathrm{d}S = \iint_{\mathcal{D}} f(G(u,v))\,\|\mathbf{n}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v$$

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(u,v))\cdot\mathbf{n}(u,v)\,\mathrm{d}u\,\mathrm{d}v$$