1. Consider the function

$$f(x,y) = 6x^3 - 6xy + y^2$$

(a) (10 points) Find the critical points of f.

Solution:

$$\nabla f = \left\langle 18x^2 - 6y, -6x + 2y \right\rangle = \langle 0, 0 \rangle$$

Solving the system of equations

$$\begin{cases} 3x^2 - y = 0 \\ -3x + y = 0 \end{cases}$$

gives the critical points $\boxed{(0,0), (1,3)}$

(b) (10 points) Describe the local behavior of f near the critical points from (a)

Solution: The discriminant is

$$\begin{aligned} D &= f_{xx} f_{yy} - (f_{xy})^2 \\ &= 36x \cdot 2 - (-6)^2 \\ &= 36(2x - 1) \end{aligned}$$

So

$$D_{(0,0)} = -36 < 0 \implies (0,0) \text{ is a saddle point}$$
$$D_{(1,3)} = 36 > 0 \text{ and } f_{xx}(1,3) = 36 > 0 \implies (1,3) \text{ is a local min}$$

2. (15 points) Use Lagrange multipliers to find the critical points of the function

$$f(x, y, z) = 2x - z + y$$

on the ellipsoid

$$x^2 + \frac{y^2}{4} + z^2 = 1.$$

Identify the global maximum and minimum values of f on the ellipsoid.

Solution:

$$\nabla f = \langle 2, 1, -1 \rangle$$
$$\nabla g = \left\langle 2x, \frac{1}{2}y, 2z \right\rangle$$

$$\nabla f = \lambda \nabla g \implies \begin{cases} 2 = 2\lambda x \\ 1 = \frac{1}{2}\lambda y \\ -1 = 2\lambda z \end{cases} \implies \begin{cases} x = \frac{1}{\lambda} \\ y = \frac{2}{\lambda} \\ z = -\frac{1}{2\lambda} \end{cases}$$

Substituting into the constraint and solving for λ:

$$\frac{1}{\lambda^2} + \frac{\frac{4}{\lambda^2}}{4} + \frac{1}{4\lambda^2} = 1 \implies \lambda = \pm\frac{3}{2}$$

At $\lambda = \frac{3}{2}$, $(x, y, z) = (\frac{2}{3}, \frac{4}{3}, -\frac{1}{3}) \implies f = 3$
At $\lambda = -\frac{3}{2}$, $(x, y, z) = (-\frac{2}{3}, -\frac{4}{3}, \frac{1}{3}) \implies f = -3$

Thus the global minimum value of f subject to the constraint is -3, and the global maximum value of f subject to the constraint is 3.

3. (15 points) Calculate the integral

$$\iiint_{\mathcal{B}} x\cos(xy) + 3z^2 \,\mathrm{d}V$$

where $\mathcal{B} = [0,\pi] \times [0,1] \times [-1,1]$.

Solution:

$$\begin{aligned}
&\int_{-1}^{1}\int_{0}^{1}\int_{0}^{\pi} x\cos(xy) + 3z^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad= \int_{-1}^{1}\int_{0}^{1}\int_{0}^{\pi} x\cos(xy)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{-1}^{1}\int_{0}^{1}\int_{0}^{\pi} 3z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad= 2\int_{0}^{\pi}\int_{0}^{1} x\cos(xy)\,\mathrm{d}y\,\mathrm{d}x + 3\pi\int_{-1}^{1} z^2\,\mathrm{d}z \\
&\quad= 2\int_{0}^{\pi}\big[\sin(xy)\big]_{y=0}^{1}\,\mathrm{d}x + \pi\big[z^3\big]_{-1}^{1} \\
&\quad= 2\int_{0}^{\pi}\sin(x)\,\mathrm{d}x + 2\pi \\
&\quad= 2\big[-\cos(x)\big]_{0}^{\pi} + 2\pi \\
&\quad= \boxed{4 + 2\pi}
\end{aligned}$$

4. (15 points) Let $\mathcal{D}$ be the region

$$x \leq 0, \quad 0 \leq y \leq 2x + 2$$

Evaluate

$$\iint_{\mathcal{D}} 6xy \, \mathrm{d}A$$

Solution:

$$\begin{aligned}\int_{-1}^{0} \int_{0}^{2x+2} 6xy \, \mathrm{d}y \, \mathrm{d}x &= \int_{-1}^{0} \left[3xy^2\right]_{y=0}^{2x+2} \mathrm{d}x \\ &= \int_{-1}^{0} 12x^3 + 24x^2 + 12x \, \mathrm{d}x \\ &= \left[3x^4 + 8x^3 + 6x^2\right]_{-1}^{0} \\ &= \boxed{-1}\end{aligned}$$

5. Consider the region $\mathcal{E}$ of points (x, y, z) satisfying

$$x^2 + y^2 + z^2 \leq 16, \quad y \leq 0, \quad z \leq 0$$

(a) (10 points) Express the triple integral,

$$\iiint_{\mathcal{E}} y \, \mathrm{d}V$$

as an iterated integral using spherical coordinates

Solution:

$$\int_{\pi}^{2\pi} \int_{\pi/2}^{\pi} \int_{0}^{4} \rho^3 \sin\theta \sin^2\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$$

(b) (5 points) Evaluate the integral (use identities on the formula sheet if needed).

Solution:

$$\int_{\pi}^{2\pi} \int_{\pi/2}^{\pi} \int_{0}^{4} \rho^3 \sin\theta \sin^2\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta = \int_0^4 \rho^3 \, \mathrm{d}\rho \cdot \int_{\pi}^{2\pi} \sin\theta \, \mathrm{d}\theta \cdot \int_{\pi/2}^{\pi} \sin^2\phi \, \mathrm{d}\phi$$

$$= 64 \cdot (-2) \cdot \frac{\pi}{4} = \boxed{-32\pi}$$

6. Let $\mathcal{R}$ be the parallelograph with vertices $(0,0), (1,1), (2,0)$ and $(3,1)$.

   (a) (5 points) Give a formula for a linear transformation $T(u,v) = (x(u,v), y(u,v))$ which maps the square $\mathcal{S} = [0,1] \times [0,1]$ onto $\mathcal{R}$.

   Solution:

   $$T(u,v) = (2u + v, v)$$

   (b) (5 points) Compute the Jacobian of T.

   Solution:

   $$Jac(T) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 2 \cdot 1 - 1 \cdot 0 = \boxed{2}$$

   (c) (10 points) Use the change of variables formula to compute the double integral

   $$\iint_{\mathcal{R}} 4ye^{x-y}\,\mathrm{d}A$$

   Solution:

   $$f(2u+v, v) = 4ve^{2u+v-v} = 4ve^{2u}$$

   so

   $$\begin{aligned}\iint_{\mathcal{R}} 4ye^{x-y}\,\mathrm{d}A &= \iint_{\mathcal{S}} 4ve^{2u} \cdot 2\,\mathrm{d}u\,\mathrm{d}v \\ &= 8\int_0^1 ve^{2u}\,\mathrm{d}u\,\mathrm{d}v \\ &= 8\int_0^1 e^{2u\,\mathrm{d}u} \cdot \int_0^1 v\,\mathrm{d}v \\ &= 8 \cdot \frac{1}{2}(e^2 - 1) \cdot \frac{1}{2} \\ &= \boxed{2(e^2 - 1)}\end{aligned}$$