Math 222 Exam 3 November 11, 2021

Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. Please explain your responses in full detail. You will have 1 hour to complete this exam.

Question	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total:	100	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. Consider the function

$$f(x, y) = 6x^3 - 6xy + y^2$$

(a) (10 points) Find the critical points of f.

(b) (10 points) Describe the local behavior of f near the critical points from (a).

2. (15 points) Use Lagrange multipliers to find the critical points of the function

$$f(x, y, z) = 2x - z + y$$

on the ellipsoid

$$x^2 + \frac{y^2}{4} + z^2 = 1.$$

Identify the global maximum and minimum values of f on the ellipsoid.

3. (15 points) Calculate the integral

$$\iiint_{\mathcal{B}} x\cos(xy) + 3z^2 \, \mathrm{d}V$$

where $\mathcal{B} = [0, \pi] \times [0, 1] \times [-1, 1]$.

4. (15 points) Let $\mathcal{D}$ be the region

$$x \leq 0, \quad 0 \leq y \leq 2x + 2$$

Evaluate

$$\iint_{\mathcal{D}} 6xy \; \mathrm{d}A$$

5. Consider the region $\mathcal{E}$ of points (x, y, z) satisfying

$$x^2 + y^2 + z^2 \leq 16, \;\; y \leq 0, \;\; z \leq 0$$

(a) (10 points) Express the triple integral,

$$\iiint_{\mathcal{E}} y \; \mathrm{d}V$$

as an iterated integral using spherical coordinates.

(b) (5 points) Evaluate the integral (use identities on the formula sheet if needed).

6. Let $\mathcal{R}$ be the parallelogram with vertices $(0,0),(1,1),(2,0)$ and $(3,1)$.

   (a) (5 points) Give a formula for a linear transformation $T(u,v) = (x(u,v), y(u,v))$ which maps the square $\mathcal{S} = [0,1] \times [0,1]$ onto $\mathcal{R}$.

   (b) (5 points) Compute the Jacobian of T.

(c) (10 points) Use the change of variables formula to compute the double integral

$$\iint_{\mathcal{R}} 4ye^{x-y}\, \mathrm{d}A$$

Derivative formulas

$$\text{Directional derivative}: \; D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u},$$
$$\text{Discriminant}: \; D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$$

Coordinate systems

Polar	Cylindrical	Spherical
$x = r\cos(\theta)$	$x = r\cos(\theta)$	$x = \rho\cos(\theta)\sin(\phi)$
$y = r\sin(\theta)$	$y = r\sin(\theta)$	$y = \rho\sin(\theta)\sin(\phi)$
	$z = z$	$z = \rho\cos(\phi)$
$r = \sqrt{x^2+y^2}$	$r = \sqrt{x^2+y^2}$	$\rho = \sqrt{x^2+y^2+z^2}$
$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$
	$z = z$	$\cot(\phi) = \dfrac{z}{\sqrt{x^2+y^2}}$
$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

Change of variables

$$T : \mathcal{S} \to \mathcal{R}$$

$$T(u,v) = (x(u,v), y(u,v))$$

$$\mathrm{Jac}(T) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{S}} f(x(u,v),y(u,v))\;|\mathrm{Jac}(T)|\;\mathrm{d}u\,\mathrm{d}v$$

Double angle formulas

$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha),\\ &= 2\cos^2(\alpha) - 1,\\ &= 1 - 2\sin^2(\alpha)\end{aligned}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$