1. Let $\mathbf{v} = \langle 2, 1, 1\rangle$ and $\mathbf{w} = \langle -1, 0, -1\rangle$.

   (a) (5 points) Compute the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

   Solution:

$$\langle 2,1,1\rangle \times \langle -1,0,-1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{vmatrix} = \langle -1,1,1\rangle$$

$$\text{area} = \| \langle -1,1,1\rangle \| = \boxed{\sqrt{3}}$$

   (b) (5 points) Is the angle θ between $\mathbf{v}$ and $\mathbf{w}$ acute, obtuse, or a right angle? Explain your response.

   Solution:

$$\mathbf{v}\cdot\mathbf{w} = -2+0-1 = -3 \implies \text{obtuse}$$

   The type of angle is classified by the dot product of the two vectors, through considering the equation

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

   (c) (5 points) Give an equation for the line passing through $(2, 1, -2)$ with direction vector $\mathbf{v}$.

   Solution:

$$\ell = \langle 2,1,-2\rangle + t\,\langle 2,1,1\rangle$$

2. Calculate the following quantities if they exist. Otherwise, explain why they do not exist. Justify either response.

(a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy - y^2}{3x^2 + 2y^2}$$

Solution: Approaching along the x-axis ($y = 0$):

$$\lim_{x\to 0} \frac{0}{3x^2} = 0$$

Approaching along the y-axis ($x = 0$):

$$\lim_{y\to 0} \frac{-y^2}{2y^2} = -\frac{1}{2}$$

The two directions give different values, so the limit does not exist.

(b) (5 points) For

$$f(x, y, z) = xe^{xy} - \cos(yz)$$

compute

$$f_{yz}(x, y, z)$$

Solution:

$$f_{yz} = f_{zy} = \frac{\partial}{\partial y}\, y\sin(yz) = \sin(yz) + y\cos(yz)z$$

(c) (5 points) For $f(x, y) = x^2 + y^2 - xy$ find the unit vector which points in the direction for which $f(x, y)$ increases the most rapidly starting at $(2, 1)$.

Solution:

$$\nabla f = \langle 2x - y, 2y - x\rangle$$
$$\nabla f(2, 1) = \langle 3, 0\rangle$$

The unit vector is $\boxed{\langle 1, 0\rangle}$.

(d) (5 points) Find the equation for the tangent plane to the surface

$$z = x^2 + y^2 - xy$$

at the point $(2, 1, 3)$.

Solution:

$$\begin{aligned} f &= x^2 + y^2 - xy - z \\ \nabla f &= \langle 2x - y, 2y - x, -1 \rangle \\ \nabla f(2,1,3) &= \langle 3, 0, -1 \rangle \\ &\boxed{0 = 3(x-2) - 1(z-3)} \end{aligned}$$

3. Let

$$f(x, y) = x^2 + 2xy - 2y$$

and $\mathcal{D}$ be the triangle in the fourth quadrant with bounds

$$x \geq 0, \quad y \leq 0, \quad y - x \geq -4$$

(a) (5 points) Find the critical points of $f(x, y)$ in the interior of $\mathcal{D}$.

Solution:

$$f_x = 2x + 2y = 0$$
$$f_y = 2x - 2 = 0$$

The solution to this system is $x = 1, y = -1$. This is the only critical point in the interior of $\mathcal{D}$.

(b) (5 points) Does $f(x, y)$ have a local max, local min or saddle point at the point(s) found in (a)? Explain your response.

Solution:

$$f_{xx} = 2$$
$$f_{yy} = 0$$
$$f_{xy} = 2$$
$$\text{disc} = 2 \cdot 0 - (2)^2 = -4$$

Since the discriminant is $-4 < 0$, the critical point is a saddle point.

(c) (5 points) Find the maximum value of $f(x, y)$ on $\mathcal{D}$.

Solution: Since the critical point previously found is a saddle point, we can ignore it. We must test the boundary of $\mathcal{D}$.

- For the side $x = 0$:

$$f(0, y) = -2y \qquad y \in [-4, 0]$$

Since $f' = -2$, there are no critical points on this side. Plugging in the end points, $f(0, 0) = 0$ and $f(0, -4) = 8$.

- For the side $y = 0$:

$$f(x, 0) = x^2 \qquad x \in [0, 4]$$

Since $f' = 2x = 0 \implies x = 0$ is a critical point. Plugging in the end points, $f(0, 0) = 0$ and $f(4, 0) = 16$.

- For the side $y = x - 4$:

$$f(x, x-4) = x^2 + 2x(x-4) - 2(x-4) = 3x^2 - 10x + 8 \qquad y \in [0, 4]$$

Since $f' = 6x - 10 = 0 \implies x = \frac{5}{3}$ is a critical point. The end points have already been computed, so we just evaluate at the critical point: $f(\frac{5}{3}, -\frac{7}{3}) = -\frac{1}{3}$.

The maximum value on $\mathcal{D}$ is thus $\boxed{16}$.

4. (10 points) Let $\mathcal{E} = [1,2] \times [-2,1] \times [0,3]$. Evaluate the triple integral

$$\iiint_{\mathcal{E}} 3z^2 - 4xy \, \mathrm{d}V$$

Solution:

$$\begin{aligned}
&\int_0^3 \int_{-2}^1 \int_1^2 3z^2 - 4xy \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&\quad = 3 \int_0^3 \int_{-2}^1 \int_1^2 z^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z - 4 \int_0^3 \int_{-2}^1 \int_1^2 xy \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&\quad = 3 \cdot 1 \cdot 3 \int_0^3 z^2 \, \mathrm{d}z - 4 \cdot 3 \int_{-2}^1 y \, \mathrm{d}y \cdot \int_1^2 x \, \mathrm{d}x \\
&\quad = 3 \left[z^3\right]_0^3 - 3 \left[y^2\right]_{-2}^1 \left[x^2\right]_1^2 \\
&\quad = 81 - 3(1-4)(4-1) = \boxed{108}
\end{aligned}$$

5. Evaluate the following integrals.

   (a) (10 points) Let $\mathcal{D}$ be the region $x^2 + y^2 \leq 9$ and $y \leq 0$. Evaluate

$$\iint_{\mathcal{D}} 2e^{x^2+y^2}\, \mathrm{d}A.$$

Solution:

$$\int_{\pi}^{2\pi} \int_0^3 2e^{r^2} \cdot r \, \mathrm{d}r \, \mathrm{d}\theta = \pi \left[e^{r^2} \right]_0^3 = \boxed{\pi(e^9 - 1)}$$

   (b) (10 points) Let $\mathcal{D}$ be the region

$$0 \leq x \leq \frac{\pi}{2}, \qquad 0 \leq y \leq \sin x.$$

Compute the integral

$$\iint_{\mathcal{D}} 2y \cos x \, \mathrm{d}A.$$

Solution:

$$\begin{aligned} &\int_0^{\pi/2} \int_0^{\sin x} 2y \cos x \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^{\pi/2} \left[y^2 \right]_0^{\sin x} \cos x \, \mathrm{d}x \\ &= \int_0^{\pi/2} \sin^2 x \cos x \, \mathrm{d}x \\ &= \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} = \boxed{\frac{1}{3}} \end{aligned}$$

6. Let

$$\mathbf{F} = \langle -2x, -2y, 4z \rangle .$$

(a) (5 points) If $\mathbf{F}$ is a conservative vector field, find a potential. Otherwise, explain why it is not conservative.

Solution: $\mathbf{F}$ is a conservative vector field with potential function

$$f(x, y, z) = -x^2 - y^2 + 2z^2.$$

(b) (5 points) Let $\mathcal{C}$ be the oriented curve with parametrization

$$\mathbf{r}(t) = \left\langle \cos\left(t^2 - t\right), e^{\sin(\pi t)}, t - 1 \right\rangle$$

for $0 \leq t \leq 1$. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Solution: By the Fundamental Theorem for Conservative Vector Fields,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\ &= f(1, 1, 0) - f(1, 1, -1) = -2 - 0 = \boxed{-2}\end{aligned}$$

(c) (5 points) Let $\mathbf{A} = \langle -2yz, 2xz, 0 \rangle$ and compute $\operatorname{curl}(\mathbf{A})$.

Solution:

$$\operatorname{curl}(\mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -2yz & 2xz & 0 \end{vmatrix} = \langle -2x, -2y, 4z \rangle$$

Note that this equals $\mathbf{F}$ given in the beginning of this problem.

(d) (5 points) Let $\mathcal{S}$ be the upper ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1, \qquad z \geq 0$$

oriented outwardly. Compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

State any theorems used in the computation.

Solution: Since $\mathbf{F} = \operatorname{curl}(\mathbf{A})$, where $\mathbf{A}$ is as in part (c), we can use Stokes theorem, which says:

$$\oint_{\partial\mathcal{S}} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{A}) \cdot \mathrm{d}\mathbf{S}$$

The boundary of $\mathcal{S}$ is the ellipse $\dfrac{x^2}{4} + y^2 = 1$. Using the parametrization $\mathbf{r}(t) = (2\cos t, \sin t, 0)$, $0 \leq t \leq 2\pi$, we can thus compute

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \oint_{\partial\mathcal{S}} \mathbf{A} \cdot \mathrm{d}\mathbf{r} \\ &= \int_0^{2\pi} \mathbf{A}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} \langle 0, 0, 0\rangle \cdot \mathbf{r}'(t)\,\mathrm{d}t = \boxed{0}\end{aligned}$$

7. Let $\mathcal{S}$ be the cone

$$x^2 + y^2 = z^2, \qquad -2 \leq z \leq 0$$

and $\mathcal{D}$ the disc

$$x^2 + y^2 \leq 4, \qquad z = -2$$

both oriented outwardly from the interior and $\mathbf{F} = \langle x, y, z \rangle$.

(a) (5 points) Compute $\operatorname{div} \mathbf{F}$.

Solution:

$$\operatorname{div} \mathbf{F} = 1 + 1 + 1 = \boxed{3}$$

(b) (5 points) Compute

$$\iint_{\mathcal{D}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Solution: Note that since $\operatorname{div} \mathbf{F} \neq 0$, $\mathbf{F}$ is not a curl vector field, hence Stokes theorem cannot be used for this problem. Instead, we must compute the vector surface integral directly: The surface $\mathcal{D}$ can be parametrized by

$$G(r, \theta) = (r\cos\theta, r\sin\theta, -2) \qquad 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi$$

The normal vector is

$$\mathbf{N}(r, \theta) = G_r \times G_\theta = \langle \cos\theta, \sin\theta, 0 \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle = \langle 0, 0, r \rangle$$

The *outward-pointing* normal vector is $\langle 0, 0, -r \rangle$. Hence

$$\begin{aligned} \iint_{\mathcal{D}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_0^{2\pi} \int_0^2 \mathbf{F}(G(r, \theta)) \cdot \mathbf{N}(r, \theta)\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^2 \langle r\cos\theta, r\sin\theta, -2 \rangle \cdot \langle 0, 0, -r \rangle\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^2 2r\, \mathrm{d}r\, \mathrm{d}\theta = \boxed{8\pi} \end{aligned}$$

(c) (5 points) The volume of a cone is $\frac{1}{3}Ah$ where h is the height of the cone and A is the area of the base. Using this and the Divergence Theorem, calculate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Solution: The divergence theorem says that

$$\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, \mathrm{d}V = \iint_{\partial \mathcal{W}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where $\mathcal{W}$ is the solid cone whose boundary is $\mathcal{S} \cup \mathcal{D}$. From part (a), we see the LHS evaluates to

$$\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, \mathrm{d}V = 3 \iiint_{\mathcal{W}} \mathrm{d}V = 3 \cdot \frac{1}{3}\pi 2^2 \cdot 2 = \boxed{8\pi}$$

From part (b), the RHS evaluates to

$$\iint_{\mathcal{D}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 8\pi + \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Hence

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \boxed{0}$$

(d) (5 points) Give another explanation of the result in (c) by calculating $\mathbf{F} \cdot \mathbf{N}$ where $\mathbf{N}$ is the orientation vector field on $\mathcal{S}$.

Solution: The surface $\mathcal{S}$ can be parametrized by

$$G(r, \theta) = (r\cos\theta, r\sin\theta, -r), \quad 0 \le r \le 2, \ 0 \le \theta \le 2\pi$$

Then

$$\mathbf{N} = G_r \times G_\theta = \langle \cos\theta, \sin\theta, -1 \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle = \langle r\cos\theta, r\sin\theta, r \rangle$$

Then $\mathbf{F} \cdot \mathbf{N} = \langle r\cos\theta, r\sin\theta, -r \rangle \cdot \langle r\cos\theta, r\sin\theta, r \rangle = 0$. Hence

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}S = 0$$