Test 2 Version A
Math 222 Fall 2022
October 13, 2022

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Time of Recitation: ______________________

Initials of Recitation Instructor: ______________

You may not use any type of calculator whatsoever. (Cell phones off and away!) You are not allowed to have any other notes, and the test is closed book. Use the backs of pages for scrapwork, and if you write anything on the back of a page which you want to be graded, then you should indicate that fact (on the front). Do not unstaple or remove pages from the exam. **Except for matters of English, the proctors will not answer any questions.**

By taking this exam you are agreeing to abide by KSU's Academic Integrity Policy.

Simple or standard simplifications should be made. You must **show your work**, and in order to get credit or partial credit, your work must make sense!

GOOD LUCK!!!

Problem	Possible	Score	Problem	Possible	Score
1	16		5	20	
2	16		6	8	
3	12		7	15	
4	13				
Total	57			43	

1. Computation:

(a) Let $f(x, y) = x^2 \exp(\overbrace{x^2 - 3xy + 2y^3}^{\text{☺}})$. Find all of the first partial derivatives. (In case you haven't seen it before, "$\exp(u)$" is the same thing as e^u.)

$$f_x = 2xe^{\text{☺}} + x^2(2x-3y)e^{\text{☺}}$$

$$= (2x^3 - 3x^2y + 2x)e^{(x^2-3xy+2y^3)}$$

$$f_y = x^2(6y^2 - 3x)e^{(x^2-3xy+2y^3)}$$

$$= 3x^2(2y^2 - x)e^{(x^2-3xy+2y^3)}$$

(b) Let $g(x, y) = \dfrac{y}{\sqrt{9x^2 + y^2}}$. Find the first partial derivative with respect to y and simplify it.

$$g_y = \frac{\sqrt{9x^2+y^2}\cdot 1 - y\cdot\frac{1}{2}(9x^2+y^2)^{-\frac{1}{2}}\cdot 2y}{9x^2+y^2}\cdot\frac{\sqrt{9x^2+y^2}}{\sqrt{9x^2+y^2}}$$

$$= \frac{9x^2+y^2-y^2}{(9x^2+y^2)^{3/2}}$$

$$= \frac{9x^2}{(9x^2+y^2)^{3/2}}$$

2. A certain differentiable function satisfies:

(a) $f(4,3) = 1$, and $f(-2,5) = -\pi^2$.

(b) $\nabla f(4,3) = (6,-7)$, and $\nabla f(-2,5) = (e^3, \sqrt{17})$.

At each of the two points in question (i.e. at $(4,3)$ and at $(-2,5)$) answer the following questions:

(a) In what direction is the function increasing the fastest and what is the rate of change in that direction?

$\langle 6,-7\rangle$ or $\dfrac{\langle 6,-7\rangle}{\sqrt{85}}$ $\qquad$ $\langle e^3, \sqrt{17}\rangle$ or $\dfrac{\langle e^3, \sqrt{17}\rangle}{\sqrt{e^6+17}}$

$\sqrt{85}$ $\qquad$ $\sqrt{e^6+17}$

(b) What is the directional derivative in the direction of the vector $\langle 3,-4\rangle$?

$\langle 6,-7\rangle \cdot \langle \frac{3}{5}, -\frac{4}{5}\rangle$ $\qquad$ $\langle e^3, \sqrt{17}\rangle \cdot \langle \frac{3}{5}, -\frac{4}{5}\rangle$

$= \dfrac{18+28}{5} = \dfrac{46}{5}$ $\qquad$ $= \dfrac{3e^3 - 4\sqrt{17}}{5}$

(c) What is the tangent plane and/or the linear approximation at each of the two points?

$z = 1 + 6(x-4) - 7(y-3)$ $\qquad$ $z = -\pi^2 + e^3(x+2) + \sqrt{17}(y-5)$

3. Set up **but do not solve** the following problems. As part of setting these problems up, you should list the unknowns and the equations that you would need to use to find them. You **should also do** all of the **derivative** calculations, but the **algebra** is totally unmanageable, so do **not** attempt it!

(a) Maximize $f(x, y) = y^4 \cos(x^3)$
Subject to $g(x, y) = x^8 + y^6 = 1$.

$\nabla f = \langle -3x^2y^4\sin(x^3), 4y^3\cos(x^3) \rangle$

$\nabla g = \langle 8x^7, 6y^5 \rangle$

3 unknowns

(x, y, λ)

$$\left.\begin{aligned} -3x^2y^4\sin(x^3) &= \lambda\cdot 8x^7 \\ 4y^3\cos(x^3) &= \lambda\cdot 6y^5 \\ x^8 + y^6 &= 1 \end{aligned}\right\} \text{3 eqn's}$$

(b) Maximize $F(x, y, z) = \cos(x^2 + y^4 + z^6)$
Subject to $G(x, y, z) = 4x + 3y + 2z = 0$
and $H(x, y, z) = x^2 + z^2 = 25$.

$\nabla F = -\sin(x^2+y^4+z^6)\cdot\langle 2x, 4y^3, 6z^5\rangle$

$\nabla G = \langle 4, 3, 2 \rangle$

$\nabla H = \langle 2x, 0, 2z \rangle$

5 unknowns

(x, y, z, λ, μ)

$$\left.\begin{aligned} -2x\sin(x^2+y^4+z^6) &= \lambda\cdot 4 + \mu\cdot 2x \\ -4y^3\sin(x^2+y^4+z^6) &= \lambda\cdot 3 \\ -6z^5\sin(x^2+y^4+z^6) &= \lambda\cdot 2 + \mu\cdot 2z \\ 4x + 3y + 2z &= 0 \\ x^2 + z^2 &= 25 \end{aligned}\right\} \text{5 eqn's}$$

4. For the function

$$f(x,y) = 4x^4 + \frac{y^4}{4} - 8xy$$

find and classify all of the critical points.

$f_x = 16x^3 - 8y = 8(2x^3 - y) = 0$

$f_y = y^3 - 8x = 0$

$y = 2x^3$

$(2x^3)^3 = 8x$

$x^9 = x$

$x = 0, \pm 1$ CPs $(0,0), (1,2), (-1,-2)$

$f_{xx} = 48x^2, \; f_{xy} = -8, \; f_{yy} = 3y^2$

$D = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 64$

$D(0,0) < 0$ Saddle

$D(1,2) > 0$ $f_{xx}(1,2) > 0$ Local Min

$D(-1,-2)$ $f_{xx}(-1,-2)$ " "

5. Find the maximum and the minimum of the function

$$f(x,y) = x^2 - 4x + y^2 + 2y$$

in the region given by

$$g(x,y) = x^2 + y^2 \leq 45.$$

Show your work carefully in this problem, and let us know what you are doing.

Step 1: Assume $g < 45$. Look for CPs.
i.e. set $\nabla f = 0$

$\nabla f = \langle 2x-4, 2y+2 \rangle = \langle 0,0 \rangle$

CP: $(2,-1)$

Step 2: Assume $g = 45$. Look for CCPs.
Use Lag. Mult's i.e. Set $\nabla f = \lambda \nabla g$

$\nabla g = \langle 2x, 2y \rangle$

$2x - 4 = \lambda \cdot 2x$
$2y + 2 = \lambda \cdot 2y$
$x^2 + y^2 = 45$

$x \neq 0, y \neq 0$ so

$\lambda = \frac{x-2}{x} = \frac{y+1}{y}$

$xy - 2y = xy + x$

$-2y = x$

$4y^2 + y^2 = 45$
$5y^2 = 45$
$y^2 = 9$
$y = \pm 3$

CCPs: $(6,-3), (-6,3)$

$f(2,-1) = 4 - 8 + 1 - 2 = -5$ Min
$f(6,-3) = 36 - 24 + 9 - 6 = 15$
$f(-6,3) = 36 + 24 + 9 + 6 = 75$ Max

6. Suppose that $x = r\cos\theta$ and $y = r\sin\theta$ (the usual polar coordinates) and $f(x, y) = xy^3$. Express

$$\frac{\partial f}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta}$$

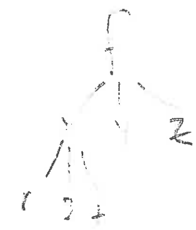
as functions of r and θ. (Hint/Comment: Do this however you like.)

$$f(x,y) = r^4 \cos\theta \sin^3\theta$$

$$f_r = 4r^3 \cos\theta \sin\theta$$

$$f_\theta = -r^4 \sin^4\theta + 3r^4 \cos^2\theta \sin^2\theta$$

$$= r^4 \sin^2\theta \left(3\cos^2\theta - \sin^2\theta\right)$$



7. Short answers ...

(a) If f is a function of x, y, and z, and x, y and z are each functions of r, s, and t, then use the chain rule to express $\dfrac{\partial f}{\partial r}$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

(b) According to the theorem that we learned, what should you require of a set S to guarantee that any continuous function f will attain an absolute maximum and an absolute minimum on S?

Closed & Bounded

(c) For the set $\overbrace{x^4 + y^3 + z^5 + 9xyz}^{F} = 6$ write down the tangent plane at the point $(-1,\ 3,\ 2)$.

$$\nabla F = \langle 4x^3 + 9yz,\ 3y^2 + 9xz,\ 5z^4 + 9xy \rangle$$

$$\nabla F(-1, 3, 2) = \langle -4 + 54,\ 27 - 18,\ 80 - 27 \rangle$$

$$= \langle 50, 9, 53 \rangle$$

$$\langle 50, 9, 53 \rangle \cdot \langle x+1,\ y-3,\ z-2 \rangle = 0$$

$$50x + 50 + 9y - 27 + 53z - 106 = 0$$

$$50x + 9y + 53z = 83$$