

Final Exam Make Up

Math 222 Fall 2022

December 14, 2022

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Recitation Time and Instructor's Initials: _____

You may not use any type of calculator whatsoever. (Cell phones off and away!) You are not allowed to have any other notes, and the test is closed book. Use the backs of pages for scrapwork, and if you write anything on the back of a page which you want to be graded, then you should indicate that fact (on the front). Except for the last page which is the **cheat sheet** (and which you should not hand in) do not unstaple or remove pages from the exam.

By taking this exam you are agreeing to abide by KSU's Academic Integrity Policy.

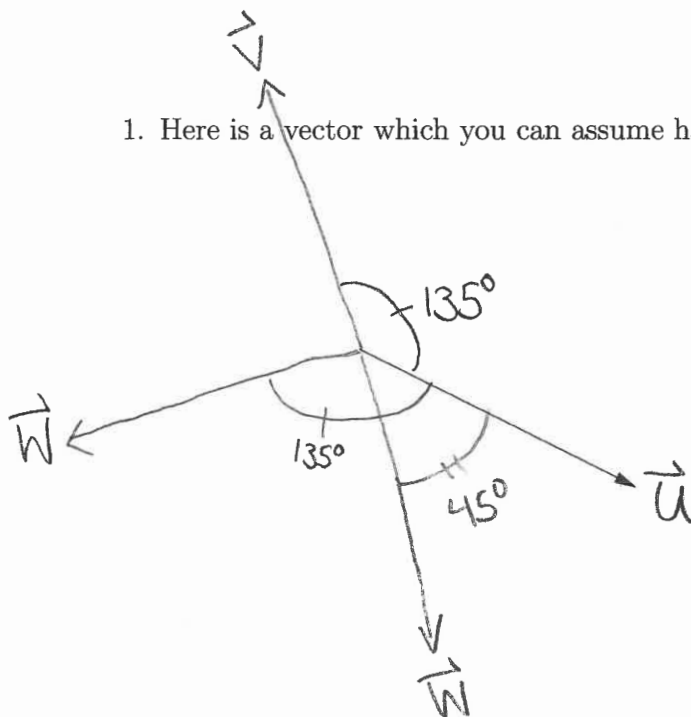
Simple or standard simplifications should be made. Box your final answers when it is reasonable. You must **show your work** for every problem, and in order to get credit or partial credit, your work must make sense!

MAY THE FORCE BE WITH YOU, YOUNG JEDI!!!

Problem	Possible	Score	Problem	Possible	Score
1	15		6	15	
2	14		7	15	
3	15		8	18	
4	16		9	12	
5	15		10	15	
Total	75			75	

Rewrite and sign the following using a pen: I give my word that I will do my best to prevent any of the information on this test leaking to anyone in this course who has yet to take it.

1. Here is a vector which you can assume has unit length:



Call this vector \vec{u} . (You should assume that this page is in a plane, and all the vectors that you draw in this problem are also assumed to be in the same plane, so there is no perspective to worry about! Stated differently, you are not drawing a **picture** of the vectors. What you draw **are** the actual vectors.)

Using the same base point draw a vector \vec{w} (and label it) so that the following are all satisfied:

- (a) $|\vec{v}| = 1$.
- (b) $\vec{u} \cdot \vec{v} \approx -\sqrt{2}/2$. (Again, do your best to get equality.)
- (c) $\vec{u} \times \vec{v}$ points toward you.

Next using the same base point again draw a vector \vec{v} (and label it) so that the following are all satisfied:

- (a) $|\vec{w}| = 1$.
- (b) $\vec{u} \times \vec{w}$ points away from you.
- (c) $|\vec{u} \times \vec{w}| \approx \sqrt{2}/2$. (Try to make it as close as you can.)

2. Short answers ... Intuition and Understanding

(a) What is the curvature of a circle with radius 49? $\frac{1}{49}$

(b) Let Ω_1 be the rectangular solid

$$\Omega_1 := \{ (x, y, z) : -1 \leq x \leq 2, 0 \leq y \leq 4, 0 \leq z \leq 3 \}.$$

and let Ω_2 be the solid cylinder

$$\Omega_2 := \{ (x, y, z) : x^2 + y^2 \leq 9, 4 \leq z \leq 7 \}$$

$$\Omega_1 \cap \Omega_2 = \emptyset$$

Assume that $f(x, y, z)$ is defined to equal 2 for all points in Ω_1 , is defined to equal -5 for all points in Ω_2 , and is defined to be 0 everywhere else. Compute:

$$\int_{x=-20}^{25} \int_{y=-25}^{20} \int_{z=-30}^{30} f(x, y, z) dz dy dx.$$

$$\begin{aligned} &= \text{Vol}(\Omega_1) \cdot 2 + \text{Vol}(\Omega_2) \cdot (-5) = 3 \cdot 4 \cdot 3 \cdot 2 - 5 \cdot \pi \cdot 3^2 \cdot 3 \\ &= 9(8 - 15\pi) = 72 - 135\pi \end{aligned}$$

(c) Will the line integral

$$\int_C f(x, y, z) dS$$

typically give you the length of the curve C ? Explain your answer in two sentences or less.

Not unless $f=1$ or by coincidence if $\text{avg } f=1$.

(d) What is the average value of the function $f(x, y) = 2x + 4y$ on the rectangle $-0 \leq x \leq 4, 1 \leq y \leq 3$?

$$I = \int_{y=1}^3 \int_{x=0}^4 (2x+4y) dx dy = \int_{y=1}^3 (x^2+4xy) \Big|_{x=0}^4 dy = \int_{y=1}^3 (16+16y) dy$$

$$= 32 + \left(8y^2 \Big|_{y=1}^3 \right) = 8 \cdot 4 + 8 \cdot (9-1) = 8 \cdot 2$$

$$\begin{aligned} \text{Avg Val.} &= \frac{8 \cdot 12}{8} \\ &= 12 \end{aligned}$$

3. Short answers ... Definitions and Theorems

- (a) Suppose that $\nabla f(0,0) = \langle 0,0 \rangle$, and $f_{xx}(0,0)$ and $f_{yy}(0,0)$ are both positive. Do you need anything else to conclude that $(0,0)$ is a local minimum? (If yes, then what exactly? If no, then why not?)

By 2ND der. test you need

$$D(0,0) := f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) > 0$$

- (b) If f is a continuous function on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1,$$

then can we know from a theorem that we learned whether or not f attains an absolute maximum and absolute minimum on that set? Why or why not?

It does. The ellipse is closed & bounded.

- (c) Assume that you have been given a differentiable vector field defined in the region $\{x^2 + y^2 + z^2 \leq 100^2\}$. How can you quickly tell if it is conservative there?

If its curl is $\langle 0,0,0 \rangle$,
then it is conservative.

4. A certain differentiable function satisfies:

(a) $f(5, -4) = 1$, and $f(-6, 3) = 2$.

(b) $\nabla f(5, -4) = \langle 9, 7 \rangle$, and $\nabla f(-6, 3) = \langle 8, -\pi \rangle$.

At each of the two points in question (i.e. at $(5, -4)$ and at $(-6, 3)$) answer the following questions:

(a) In what direction is the function increasing the fastest?

or $\frac{\langle 9, 7 \rangle}{\sqrt{130}}$

or $\frac{\langle 8, -\pi \rangle}{\sqrt{64 + \pi^2}}$

(b) What is the rate of change in that direction?

$\sqrt{130}$

$\sqrt{64 + \pi^2}$

(c) What is the directional derivative in the direction of the vector $\langle 3, -4 \rangle$? (Note: I did not ask for the directional derivative in the direction of the point $(3, -4)$.)

$$\begin{aligned} D_{\vec{e}} f &= \nabla f \cdot \vec{e} \\ &= \langle 9, 7 \rangle \cdot \frac{\langle 3, -4 \rangle}{5} \\ &= -\frac{1}{5} \end{aligned}$$

$$\begin{aligned} D_{\vec{e}} f &= \nabla f \cdot \vec{e} \\ &= \langle 8, -\pi \rangle \cdot \frac{\langle 3, -4 \rangle}{5} \\ &= \frac{24 + 4\pi}{5} = \frac{4}{5}(6 + \pi) \end{aligned}$$

(d) What is the tangent plane and/or the linear approximation at each of the two points?

$$Z = 1 + 9(x-5) + 7(y+4)$$

$$Z = 2 + 8(x+6) - \pi(y-3)$$

5. Find the maximum and minimum of the function

$$f(x, y) = 2x - x^2 - 4y - y^2$$

on the set

$$g(x, y) = x^2 + y^2 \leq 80.$$

Show your work, and give a short explanation of what you are doing.

(No essays, please. Just a few short words in the right places will suffice.)

Step 1: Assume $g < 80$. Look for CPs. Set $\nabla f = \vec{0}$.

$$\nabla f = \langle 2 - 2x, -4 - 2y \rangle \quad \text{CP: } (1, -2)$$

Step 2: Assume $g = 80$. Look for CCPs. Use Lag.

Mult's Set $\nabla f = \lambda \nabla g$

$$x \neq 0 \quad y \neq 0$$

$$\nabla g = \langle 2x, 2y \rangle$$

$$\frac{1-x}{x} = \frac{-2-y}{y} = \lambda$$

$$2(1-x) = 2x\lambda$$

$$y - xy = -2x - xy$$

$$2(-2-y) = 2y\lambda$$

$$y = -2x$$

$$x^2 + y^2 = 80$$

$$x^2 + 4x^2 = 5 \cdot 16$$

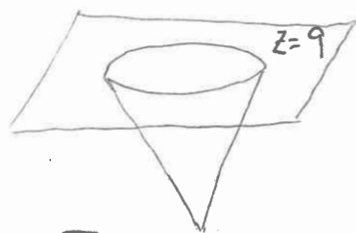
$$x^2 = 16$$

$$\text{CCPs: } (4, -8), (-4, 8)$$

$$f(1, -2) = 2 - 1 + 8 - 4 = 5 \text{ Max}$$

$$f(4, -8) = 8 - 16 + 32 - 64 = -40$$

$$f(-4, 8) = -8 - 16 - 32 - 64 = -120 \text{ Min}$$



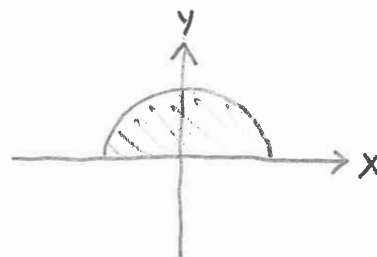
Pic Ignoring $y \geq 0$

6. Let S be the part of the set

$$z = \sqrt{x^2 + y^2}$$

with $z \leq 9$ and $y \geq 0$.

Express the integral of $f(x, y, z) = y^3$ over S as an iterated integral (i.e. a double or triple integral) over a subset of \mathbb{R}^2 or \mathbb{R}^3 which has constant bounds of integration. You do NOT need to find this integral.



Pic from above

$$\vec{r}(u, v) = (\underbrace{u}_{\hat{i}} \cos v, \underbrace{u}_{\hat{j}} \sin v, \underbrace{u}_{\hat{k}})$$

$$0 \leq u \leq 9, \quad 0 \leq v \leq \pi$$

$$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$$

$$\vec{r}_v = \langle -u \sin v, u \cos v, 0 \rangle$$

$$\vec{N} = \langle -u \cos v, -u \sin v, u \rangle$$

$$|\vec{N}| = \sqrt{u^2 + u^2} = u\sqrt{2}$$

$$\iint_S y^3 dS = \int_{u=0}^9 \int_{v=0}^{\pi} u^3 \sin^3 v \cdot u\sqrt{2} \, dv \, du$$

$$= \sqrt{2} \int_{u=0}^9 \int_{v=0}^{\pi} u^4 \sin^3 v \, dv \, du$$

7. Let C be the "horizontal zig-zag" given by the (finite) sequence of line segments that goes from $(1, 1)$ to $(3, -1)$ to $(5, 1)$ to $(7, -1)$ to $(9, 1)$ to $(11, -1)$ and finally to $(13, 1)$ where it stops. Let

$$\vec{F}(x, y) := \langle 5y^3 e^{5x}, 3y^2 e^{5x} \rangle.$$

Compute

$$I = \int_C \vec{F} \cdot d\vec{r}.$$

$$F_y^1 = 15y^2 e^{5x} \equiv F_x^2 = 15y^2 e^{5x}$$

$\uparrow \Rightarrow \vec{F} \text{ is conserv.}$

$$\vec{F} = \nabla f$$

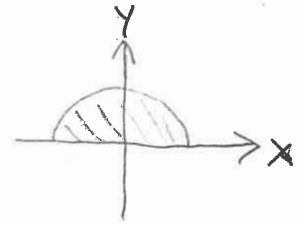
$$f = \int 5y^3 e^{5x} dx = y^3 e^{5x} + \overline{C}(y)$$

$$f = \int 3y^2 e^{5x} dy = y^3 e^{5x} + \hat{C}(x)$$

$$\vec{F} = \nabla (y^3 e^{5x})$$

$$I = y^3 e^{5x} \Big|_{(1,1)}^{(13,1)} = 1 \cdot e^{65} - 1 \cdot e^5 = e^{65} - e^5 = e^5 (e^{60} - 1)$$

F.T.L.I.



8. Let Q be the set:

$$\{ (x, y, z) : x^2 + y^2 + z^2 \leq 16, \text{ and } y \geq 0 \}$$

and let ∂Q be the boundary of this set. Let

$$\vec{F}(x, y, z) = \langle \log(2 + y^4 + \cos z), y^2, x^2 e^{3y} \rangle.$$

If \vec{n} is the outward unit normal to this region, then find

$$\iint_{\partial Q} \vec{F} \cdot \vec{n} dS = \iiint_Q \nabla \cdot \vec{F} dV = I$$

Awful w/o Div Thm $\nabla \cdot \vec{F} = 2y$

$$I = \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi} \int_{\rho=0}^4 2\rho \sin\theta \sin\phi \cdot \rho^2 \sin\phi d\rho d\theta d\phi$$

$$= 2 \int_{\rho=0}^4 \rho^3 d\rho \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{\pi} \sin^2\phi d\phi$$

$$= 2 \cdot \frac{4^4}{4} \cdot 2 \cdot \int_0^{\pi} \frac{1 - \cos 2\phi}{2} d\phi$$

$$= \frac{4^4 \pi}{2} = 2^7 \pi = 128\pi$$

9. Let C be the piece of the graph of $y = x^2$ which is between $x = 0$ and $x = 3$. Let

$$\vec{F} := \langle 3y, 1 \rangle$$

and compute

$$I = \int_C \vec{F} \cdot d\vec{r}.$$

$$F_y' = 3 \neq 0 = F_x'$$

NOT
conserv.

$$\vec{r}(t) = (t, t^2), \quad 0 \leq t \leq 3$$

$$\vec{r}'(t) = \langle 1, 2t \rangle$$

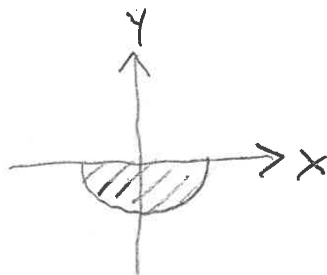
$$I = \int_{t=0}^3 \langle 3t^2, 1 \rangle \cdot \langle 1, 2t \rangle dt$$

$$= \int_0^3 (3t^2 + 2t) dt$$

$$= (t^3 + t^2) \Big|_0^3$$

$$= 27 + 9$$

$$= 36$$



10. Let E be the set

$$\{(x, y, z) : x^2 + y^2 \leq z \leq 4, y \leq 0\}.$$

Find

$$\iiint_E y^2 dV.$$

Cyl. Coords!

$$I = \int_{\theta=\pi}^{2\pi} \int_{r=0}^2 \int_{z=r^2}^4 r^2 \sin^2 \theta \, dz \, r \, dr \, d\theta$$

$$= \int_{\theta=\pi}^{2\pi} \sin^2 \theta \, d\theta \int_{r=0}^2 r^3 (4 - r^2) \, dr$$

$$= \int_{\pi}^{2\pi} \frac{1 - \cos 2\theta}{2} \, d\theta \int_0^2 (4r^3 - r^5) \, dr$$

$$= \frac{\pi}{2} \cdot \left(r^4 - \frac{r^6}{6} \right) \Big|_0^2$$

$$= \frac{\pi}{2} \left(\frac{2^5}{2} - \frac{2^6}{6} \right) = \frac{2^4 \pi}{6}$$

$$= \frac{2^3 \pi}{3} = \frac{8\pi}{3}$$

$$I = \int_{\theta=\pi}^{2\pi} \int_{z=0}^4 \int_{r=0}^{\sqrt{z}} r^2 \sin^2 \theta \, r \, dr \, dz \, d\theta$$

$$= \int_{\theta=\pi}^{2\pi} \sin^2 \theta \, d\theta \int_{z=0}^4 \left. \frac{r^4}{4} \right|_{r=0}^{\sqrt{z}} dz$$

$$= \int_{\pi}^{2\pi} \frac{1 - \cos 2\theta}{2} \, d\theta \int_0^4 \frac{z^2}{4} \, dz$$

$$= \frac{\pi}{2} \cdot \frac{4^3}{12} = \frac{\pi \cdot 4^2}{2 \cdot 3} = \frac{8\pi}{3}$$

Integral Definitions and Basic Formulas:

Line Integrals:

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}'(t) dt, \quad \text{Orientation Matters!}$$

$$\int_C f(\vec{r}) ds = \int_a^b f(\vec{r}(t)) ||\vec{r}'(t)|| dt, \quad \text{Orientation Doesn't Matter!}$$

$$\int_C f(\vec{r}) dx = \int_a^b f(x(t), y(t), z(t)) x'(t) dt, \quad \text{Orientation Matters!}$$

Surface Integrals: With $\vec{N} := \vec{r}_u \times \vec{r}_v \neq 0$, and with $\vec{r}(R) = S$ we have

$$\int \int_S \vec{F} \bullet \vec{n} dS = \int \int_R \vec{F}(\vec{r}(u, v)) \bullet \vec{N}(u, v) du dv \quad \text{Orientation Matters!}$$

$$\int \int_S f(\vec{r}) dS = \int \int_R f(\vec{r}(u, v)) ||\vec{N}(u, v)|| du dv \quad \text{Orientation Doesn't Matter!}$$

Spherical Coordinates:

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

$$dV = \rho^2 \sin \phi d\rho d\phi d\theta.$$

Second Derivative Test: Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose $\nabla f(a, b) = (0, 0)$. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2.$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum.
- (c) If $D < 0$, then (a, b) is a saddle point.

Cheat Sheet Bonus:

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}, \quad \sin^2 \theta = \frac{1 - \cos(2\theta)}{2}, \quad \sin(2\theta) = 2 \sin \theta \cos \theta.$$