Final Exam Version A
Math 222 Fall 2022
December 14, 2022
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Recitation Time and Instructor's Initials: ____________________

You may not use any type of calculator whatsoever. (Cell phones off and away!) You are not allowed to have any other notes, and the test is closed book. Use the backs of pages for scrapwork, and if you write anything on the back of a page which you want to be graded, then you should indicate that fact (on the front). Except for the last page which is the **cheat sheet** (and which you should not hand in) do not unstaple or remove pages from the exam.

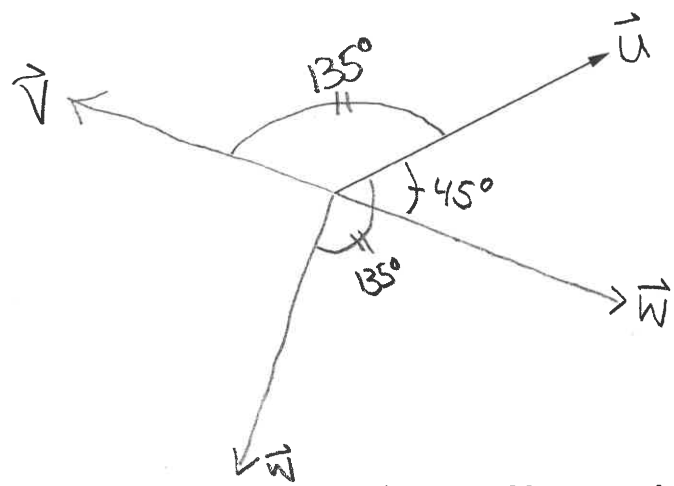
By taking this exam you are agreeing to abide by KSU's Academic Integrity Policy.

Simple or standard simplifications should be made. Box your final answers when it is reasonable. You must **show your work** for every problem, and in order to get credit or partial credit, your work must make sense!

MAY THE FORCE BE WITH YOU, YOUNG JEDI!!!

Problem	Possible	Score	Problem	Possible	Score
1	15		6	15	
2	14		7	15	
3	15		8	18	
4	16		9	12	
5	15		10	15	
Total	75			75	

1. Here is a vector which you can assume has unit length:



Call this vector $\vec{u}$. (You should assume that this page is in a plane, and all the vectors that you draw in this problem are also assumed to be in the same plane, so there is no perspective to worry about! Stated differently, you are not drawing a **picture** of the vectors. What you draw **are** the actual vectors.)

Using the same base point draw a vector $\vec{w}$ (and label it) so that the following are all satisfied:

(a) $|\vec{w}| = 1$.

(b) $\vec{u} \times \vec{w}$ points away from you.

(c) $|\vec{u} \times \vec{w}| \approx \sqrt{2}/2$. (Try to make it as close as you can.)

Next using the same base point again draw a vector $\vec{v}$ (and label it) so that the following are all satisfied:

(a) $|\vec{v}| = 1$.

(b) $\vec{u} \bullet \vec{v} \approx -\sqrt{2}/2$. (Again, do your best to get equality.)

(c) $\vec{u} \times \vec{v}$ points toward you.

2. Short answers ... Intuition and Understanding

(a) What is the curvature of a circle with radius 16? $\frac{1}{16}$

(b) Let Ω_1 be the solid cylinder

$$\Omega_1 := \{\ (x, y, z)\ :\ x^2 + y^2 \leq 4,\ 3 \leq z \leq 7\ \}$$

and let Ω_2 be the cube

$\Omega_1 \cap \Omega_2 = \phi$

$$\Omega_2 := \{\ (x, y, z)\ :\ 0 \leq x \leq 2,\ 0 \leq y \leq 2,\ 0 \leq z \leq 2\ \}.$$

Assume that $f(x, y, z)$ is defined to equal 3 for all points in Ω_1, is defined to equal -4 for all points in Ω_2, and is defined to be 0 everywhere else. Compute:

$$\int_{x=-20}^{25} \int_{y=-25}^{20} \int_{z=-30}^{30} f(x, y, z)\ dz\ dy\ dx\ .$$

$$= \mathrm{Vol}(\Omega_1)\cdot 3 + \mathrm{Vol}(\Omega_2)\cdot(-4) = (\pi\cdot 2^2\cdot 4)\cdot 3 + 2^3\cdot(-4)$$
$$= 16(3\pi - 2) = 48\pi - 32$$

(c) Will the line integral

$$\int_C f(x, y, z)\ ds$$

typically give you the length of the curve C? Explain your answer in two sentences or less.

Not unless $f \equiv 1$ or by coincidence if $\mathrm{Avg}\, f = 1$.

(d) What is the average value of the function $f(x, y) = 2x + 4y$ on the rectangle $-1 \leq x \leq 5,\ 2 \leq y \leq 4$?

$$I = \int_{y=2}^{4}\int_{x=-1}^{5} (2x+4y)\,dx\,dy = \int_{y=2}^{4} (x^2+4xy)\Big|_{x=-1}^{5}\,dy$$

$$= \int_{y=2}^{4} [(25+20y)-(1-4y)]\,dy = \int_{y=2}^{4}(24+24y)\,dy = 48 + \left(12y^2\Big|_2^4\right)$$
$$= 48 + 144 = 192$$

$$\boxed{\mathrm{Avg\ Val} = \frac{192}{12} = 16}$$

3. Short answers ... Definitions and Theorems

(a) Suppose that $\nabla f(0,0) = \langle 0,0 \rangle$, and $f_{xx}(0,0)$ and $f_{yy}(0,0)$ are both positive. Do you need anything else to conclude that $(0,0)$ is a local minimum? (If yes, then what exactly? If no, then why not?)

By 2^{ND} der. test you need

$D(0,0) := f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) > 0$

(b) If f is a continuous function on the circle $(x-1)^2 + (y-2)^2 = 7^2$, then can we know from a theorem that we learned whether or not f attains an absolute maximum and absolute minimum on that set? Why or why not?

It does. The circle is closed & bounded.

(c) Assume that you have been given a differentiable vector field defined in the region $\{z > 0\}$. How can you quickly tell if it is conservative there?

If its curl is $\equiv \langle 0,0,0 \rangle$, then it is conservative.

4. A certain differentiable function satisfies:

(a) $f(5,-3) = 1$, and $f(-6,4) = 2$.

(b) $\nabla f(5,-3) = \langle 9,7\rangle$, and $\nabla f(-6,4) = \langle 8,-\pi\rangle$.

At each of the two points in question (i.e. at $(5,-3)$ and at $(-6,4)$) answer the following questions:

(a) In what direction is the function increasing the fastest?

$\langle 9,7\rangle$ or $\dfrac{\langle 9,7\rangle}{\sqrt{81+49}} = \dfrac{\langle 9,7\rangle}{\sqrt{130}}$

$\langle 8,-\pi\rangle$ or $\dfrac{\langle 8,-\pi\rangle}{\sqrt{64+\pi^2}}$

(b) What is the rate of change in that direction?

$\sqrt{130}$

$\sqrt{64+\pi^2}$

(c) What is the directional derivative in the direction of the **vector** $\langle 3,-4\rangle$? (Note: I did not ask for the directional derivative in the direction of the **point** $(3,-4)$.)

$$D_{\vec{e}}f = \nabla f \cdot \vec{e} = \langle 9,7\rangle \cdot \frac{\langle 3,-4\rangle}{5} = -\frac{1}{5}$$

$$D_{\vec{e}}f = \nabla f \cdot \vec{e} = \langle 8,-\pi\rangle \cdot \frac{\langle 3,-4\rangle}{5} = \frac{24+4\pi}{5} = \frac{4}{5}(6+\pi)$$

(d) What is the tangent plane and/or the linear approximation at each of the two points?

$$z = 1 + 9(x-5) + 7(y+3)$$

$$z = 2 + 8(x+6) - \pi(y-4)$$

5. Find the maximum and minimum of the function

$$f(x,y) = 6x - x^2 - 4y - y^2$$

on the set

$$g(x,y) = x^2 + y^2 \leq 117 = 13 \cdot 9.$$

Show your work, and give a short explanation of what you are doing. (No essays, please. Just a few short words in the right places will suffice.)

Step 1: Assume $g < 117$. Look for CPs

Set $\nabla f = \vec{0}$

$\nabla f = \langle 6-2x, -4-2y \rangle$

CP: $(3,-2)$

Step 2: Assume $g = 117$. Look for CCPs

Use Lag. Mult.'s. Set $\nabla f = \lambda \nabla g$.

$\nabla g = \langle 2x, 2y \rangle$

$2(3-x) = 2x\lambda$

$2(-2-y) = 2y\lambda$

$x^2 + y^2 = 13 \cdot 9$

$x \neq 0,\ y \neq 0$

$\frac{3-x}{x} = \frac{-2-y}{y} = \lambda$

$3y - xy = -2x - xy$

$3y = -2x$

$y = -\frac{2}{3}x$

$\frac{9}{9}x^2 + \frac{4}{9}x^2 = 13 \cdot 9$

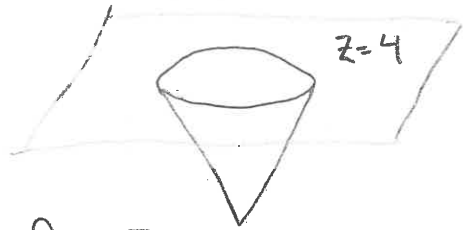
$\Rightarrow x^2 = 81$

CCPs: $(9,-6), (-9,6)$

$f(3,-2) = 18 - 9 + 8 - 4 = 9 + 4 = 13$ Max

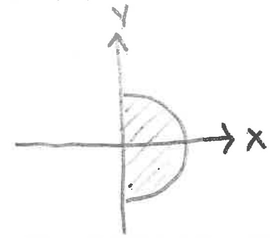
$f(9,-6) = 54 - 81 + 24 - 36 = -39$

$f(-9,6) = -54 - 81 - 24 - 36 = -195$ Min



Pic Ignoring $x \geq 0$

Pic from above



6. Let S be the part of the set

$$z = \sqrt{x^2 + y^2}$$

with $z \leq 4$ and $x \geq 0$.

Express the integral of $f(x, y, z) = x^2$ over S as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ which has **constant** bounds of integration. You do **NOT** need to find this integral.

$$\vec{r}(u,v) = (u\cos v, u\sin v, u) \qquad 0 \leq u \leq 4, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$$

$$\hat{i} \quad \hat{j} \quad \hat{k}$$

$$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{N} = \langle -u\cos v, -u\sin v, u \rangle$$

$$|\vec{N}| = \sqrt{u^2 + u^2} = u\sqrt{2}$$

$$\iint_S x^2 \, dS = \int_{u=0}^{4} \int_{v=-\frac{\pi}{2}}^{\frac{\pi}{2}} u^2\cos^2 v \cdot u\sqrt{2} \, dv \, du$$

$$= \sqrt{2} \int_{u=0}^{4} \int_{v=-\frac{\pi}{2}}^{\frac{\pi}{2}} u^3 \cos^2 v \, dv \, du$$

7. Let C be the "horizontal zig-zag" given by the (finite) sequence of line segments that goes from $(1,1)$ to $(2,0)$ to $(3,1)$ to $(4,0)$ to $(5,1)$ to $(6,0)$ and finally to $(7,1)$ where it stops. Let

$$\vec{F}(x,y) := \langle 2y^2e^{2x}, 2ye^{2x} \rangle \, .$$

Compute

$$I = \int_C \vec{F} \bullet d\vec{r} \, .$$

$$F^1_y = 4ye^{2x} \equiv F^2_x = 4ye^{2x}$$

$\Rightarrow \vec{F}$ is conservative

$$\vec{F} = \nabla f$$

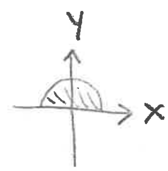
$$f = \int 2y^2 e^{2x}\, dx = y^2 e^{2x} + \bar{C}(y)$$

$$f = \int 2ye^{2x}\, dy = y^2 e^{2x} + \hat{C}(x)$$

$$\vec{F} = \nabla (y^2 e^{2x})$$

$$I = y^2 e^{2x} \Big|_{(1,1)}^{(7,1)} = 1e^{14} - 1e^{2} = e^{14} - e^{2}$$
$$= e^{2}(e^{12} - 1)$$

F.T.L.I.



8. Let Q be the set:

$$\{\, (x,y,z) \; : \; x^2+y^2+z^2 \le 9, \text{ and } y \ge 0 \,\}$$

and let ∂Q be the boundary of this set. Let

$$\vec{F}(x,y,z) = \langle\, \log(2+y^4+\cos z),\; y^2,\; x^2e^{3y} \,\rangle \, .$$

If $\vec{n}$ is the outward unit normal to this region, then find

$$\iint_{\partial Q} \vec{F}\cdot\vec{n}\, dS \underset{\uparrow}{=} \iiint_Q \nabla\cdot\vec{F}\, dV = I$$

Awful w/o DivThm.

$$\nabla\cdot\vec{F} = 2y$$

$$I = \int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\rho=0}^{3} 2\rho\sin\theta\sin\phi\cdot\rho^2\sin\phi \; d\rho\, d\theta\, d\phi$$

$$= 2\int_{\rho=0}^{3}\rho^3\, d\rho \int_{\theta=0}^{\pi}\sin\theta\, d\theta \int_{\phi=0}^{\pi}\sin^2\phi\, d\phi$$

$$= 2\cdot\frac{3^4}{4}\cdot 2\cdot\int_0^{\pi}\frac{1-\cancel{\cos 2\phi}^{\,0}}{2}\, d\phi$$

$$= \frac{3^4\pi}{2} = \frac{81\pi}{2}$$

9. Let C be the piece of the graph of $y = x^2$ which is between $x = 0$ and $x = 2$. Let

$$\vec{F} := \langle\, 3y,\ 1 \,\rangle$$

and compute

$$I = \int_C \vec{F} \bullet d\vec{r}\,.$$

$\vec{r}(t) = (t, t^2)$

$0 \le t \le 2$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$F^1_y = 3 \neq 0 = F^2_x$

NOT Conserv.

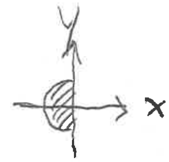
$$I = \int_{t=0}^{2} \langle 3t^2, 1 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_{t=0}^{2} (3t^2 + 2t)\, dt$$

$$= (t^3 + t^2)\Big|_0^2$$

$$= 8 + 4$$

$$= 12$$



10. Let E be the set

$$\{\,(x,y,z)\;:\;x^2+y^2\le z\le 4,\;x\le 0\,\}.$$

Find

$$I=\iiint_E x\,dV.$$ Cyl. Coords!

$$I=\int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{r=0}^{2}\int_{z=r^2}^{4} r\cos\theta\,dz\,r\,dr\,d\theta$$

$$=\int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\theta\,d\theta\int_{r=0}^{2} r^2\cdot(4-r^2)\,dr$$

$$=-2\int_{r=0}^{2}(4r^2-r^4)\,dr$$

$$=-2\left(4\frac{r^3}{3}-\frac{r^5}{5}\right)\Big|_0^2$$

$$=2^6\left(\frac{1}{5}-\frac{1}{3}\right)=\frac{-2^7}{15}$$

$$I=\int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{z=0}^{4}\int_{r=0}^{\sqrt{z}} r\cos\theta\cdot r\,dr\,dz\,d$$

$$=\int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\theta\,d\theta\int_{z=0}^{4}\left(\frac{r^3}{3}\Big|_{r=0}^{z^{1/2}}\right)dz$$

$$=-\frac{2}{3}\int_0^4 z^{3/2}\,dz$$

$$=-\frac{2}{3}\cdot\frac{2}{5}z^{5/2}\Big|_0^4$$

$$=\frac{-2^2}{15}\cdot 2^5=\frac{-2^7}{15}$$

Integral Definitions and Basic Formulas:

Line Integrals:

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}'(t)\, dt\ , \qquad \text{Orientation Matters!}$$

$$\int_C f(\vec{r})\, ds = \int_a^b f(\vec{r}(t))\, ||\vec{r}'(t)||\, dt\ , \qquad \text{Orientation Doesn't Matter!}$$

$$\int_C f(\vec{r})\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt\ , \qquad \text{Orientation Matters!}$$

Surface Integrals: With $\vec{N} := \vec{r}_u \times \vec{r}_v \neq 0$, and with $\vec{r}(R) = S$ we have

$$\int\int_S \vec{F} \bullet \vec{n}\, dS = \int\int_R \vec{F}(\vec{r}(u,v)) \bullet \vec{N}(u,v)\, du\, dv \quad \text{Orientation Matters!}$$

$$\int\int_S f(\vec{r})\, dS = \int\int_R f(\vec{r}(u,v))\, ||\vec{N}(u,v)||\, du\, dv \quad \text{Orientation Doesn't Matter!}$$

Spherical Coordinates:

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho\cos\phi$$
$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta\ .$$

Second Derivative Test: Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose $\nabla f(a,b) = (0,0)$. Let

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2\ .$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.

(c) If $D < 0$, then (a,b) is a saddle point.

Cheat Sheet Bonus:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2},\ \sin^2\theta = \frac{1-\cos(2\theta)}{2},\ \sin(2\theta) = 2\sin\theta\cos\theta\ .$$