1. (10 points) §2.1: Let $P = (0, 2)$, $Q = (1, 3)$, $R = (-4, -6)$.

   (a) Find and express in both component form and by using standard unit vectors:

$$\overrightarrow{PQ},\ \overrightarrow{PR},\ \overrightarrow{PQ} - \overrightarrow{PR}$$

Solution:

$$\overrightarrow{PQ} = \boxed{\langle 1, 1\rangle = \mathbf{i} + \mathbf{j}}$$
$$\overrightarrow{PR} = \boxed{\langle -4, -8\rangle = -4\mathbf{i} - 8\mathbf{j}}$$
$$\overrightarrow{PQ} - \overrightarrow{PR} = \langle 1, 1\rangle - \langle -4, -8\rangle = \boxed{\langle 5, 9\rangle = 5\mathbf{i} + 9\mathbf{j}}$$

   (b) Find a unit vector in direction $\overrightarrow{PR}$.

Solution:

$$\begin{aligned}\frac{\overrightarrow{PR}}{\|\overrightarrow{PR}\|} &= \frac{\langle -4, -8\rangle}{\sqrt{16 + 64}} \\ &= \frac{1}{\sqrt{80}}\langle -4, -8\rangle \\ &= \frac{1}{4\sqrt{5}}\langle -4, -8\rangle \\ &= \boxed{\left\langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle}\end{aligned}$$

   (c) Find $\mathbf{v}$ with the same direction as $\overrightarrow{PQ} - \overrightarrow{PR}$ and length 10.

Solution:

$$\begin{aligned}10 \cdot \frac{\overrightarrow{PQ} - \overrightarrow{PR}}{\|\overrightarrow{PQ} - \overrightarrow{PR}\|} &= 10\frac{\langle 5, 9\rangle}{\sqrt{25 + 81}} \\ &= \frac{10}{\sqrt{106}}\langle 5, 9\rangle \\ &= \boxed{\left\langle \frac{50}{\sqrt{106}}, \frac{90}{\sqrt{106}} \right\rangle}\end{aligned}$$

2. (10 points) §2.2:

   (a) Give an equation for S, the set of points of distance 5 from the point $C = (2, -4, -6)$ in $\mathbb{R}^3$.

   Solution:
   $$S := \boxed{(x-2)^2 + (y+4)^2 + (z+6)^2 = 25}$$

   (b) Give an equation to describe the intersection of S and the plane $y = -3$.

   Solution: Setting $y = -3$ to the description for S determined in the previous part,
   $$(x-2)^2 + (-3+4)^2 + (z+6)^2 = 25$$
   $$\boxed{(x-2)^2 + (z+6)^2 = 24}$$

   (c) Given $P = (2, -9, -6)$ in S, find the unique point Q of distance 10 from P which is also in S.

   Solution: The sphere is centered at $C = (2, -4, -6)$ and of radius 5. The given point $P = (2, -9, -6)$ is therefore the left-most point (most negative in the y-direction) on the sphere. The point Q we are looking for is thus the right-most point (most positive in the y-direction): $\boxed{(2, 1, -6)}$

   (d) Find a unit-length vector in the direction of $\overrightarrow{PQ}$.

   Solution: $\overrightarrow{PQ} = \langle 0, 10, 0 \rangle$, so $\boxed{\langle 0, 1, 0 \rangle}$

3. (10 points) §2.3: Let $\mathbf{u} = \langle 1, 4, -3 \rangle$ and $\mathbf{v} = \langle 2, -5, 2 \rangle$.

(a) Find $\mathbf{u} \cdot \mathbf{v}$. Find the angle $\theta \in [0, \pi]$ between $\mathbf{u}, \mathbf{v}$.

Solution: $\mathbf{u} \cdot \mathbf{v} = 2 - 20 - 6 = -24$. Using the formula

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|},$$

and computing

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{4 + 25 + 4} = \sqrt{33} \\ \|\mathbf{u}\| &= \sqrt{1 + 16 + 9} = \sqrt{26}, \end{aligned}$$

we obtain

$$\cos\theta = \frac{-24}{\sqrt{26 \cdot 33}} \implies \boxed{\theta = \arccos\left(\frac{-24}{\sqrt{858}}\right)}$$

(b) Find $\left\|\text{proj}_{\mathbf{u}}(\mathbf{v})\right\|$. Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$

Solution:

$$\begin{aligned} \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \\ &= \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= -\frac{24}{26}\langle 1, 4, -3 \rangle \\ &= -\frac{12}{13}\langle 1, 4, -3 \rangle = \boxed{\left\langle -\frac{12}{13}, -\frac{48}{13}, \frac{36}{13} \right\rangle} \end{aligned}$$

and

$$\left\|\text{proj}_{\mathbf{u}}(\mathbf{v})\right\| = \left|\frac{-12}{13}\right| \sqrt{1 + 16 + 9} = \boxed{\frac{12}{13}\sqrt{26}}$$

(c) Are $\mathbf{v}$ and the standard basis vector $\mathbf{k}$ orthogonal? Why or why not?

Solution:

$$\mathbf{v} \cdot \mathbf{k} = \langle 2, -5, 2 \rangle \cdot \langle 0, 0, 1 \rangle = 2 \neq 0$$

so NO, $\mathbf{v}$ and $\mathbf{k}$ are not orthogonal.

(d) Find the work done by a force $\mathbf{F} = \langle 1, -4, -3 \rangle$ applied to move an object in a straight line from the terminal point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$. Let units be in feet and lbs.

Solution:

$$\begin{aligned} \mathbf{W} &= \mathbf{F} \cdot \overrightarrow{uv} \text{ where } \overrightarrow{uv} = \langle 2-1, -5-4, -2+3 \rangle = \langle 1, -9, 5 \rangle \\ &= \langle 1, -4, -3 \rangle \cdot \langle 1, -9, 5 \rangle \\ &= 1 + 36 - 15 = 22 \text{ ft} \cdot \text{lbs.} \end{aligned}$$

4. (10 points) §2.4: Let $\mathbf{u} = \langle 4, 1, 3\rangle$ and $\mathbf{v} = \langle -5, 1, 6\rangle$.

(a) Find $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$.

Solution:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 3 \\ -5 & 1 & 6 \end{vmatrix} \\ &= 3\mathbf{i} - 39\mathbf{j} + 9\mathbf{k} \\ &= \langle 3, -39, 9\rangle\end{aligned}$$

with

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{9 + 39^2 + 81} = \sqrt{1611} \approx 40.13726.$$

(b) Find a unit-length vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{\langle 3, -39, 9\rangle}{\sqrt{1611}}.$$

(c) Let $P = \langle 2, -5, 7\rangle$. Find the area of the triangle spanned by P and the terminal points of $\mathbf{u}$, $\mathbf{v}$

Solution: Let $P = \langle 2, -5, 7\rangle$. Then

$$\text{Area}(\Delta Pvu) = \frac{1}{2}\|\overrightarrow{Pv} \times \overrightarrow{Pu}\|$$

with $u = (4, 1, 3)$, $v = (-5, 1, 6)$ gives $\overrightarrow{Pv} = \langle -7, 6, -1\rangle$, $\overrightarrow{Pu} = \langle 2, 6, -4\rangle$ and

$$\begin{aligned}\overrightarrow{Pv} \times \overrightarrow{Pu} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 6 & -1 \\ 2 & 6 & -4 \end{vmatrix} \\ &= -18\mathbf{i} - 30\mathbf{j} - 54\mathbf{k}\end{aligned}$$

The area of the triangle is thus

$$\frac{1}{2}||\overrightarrow{Pv} \times \overrightarrow{Pu}\| = \frac{1}{2}\sqrt{18^2 + 30^2 + 54^2} = \boxed{\frac{1}{2}\sqrt{4140} \approx 32.1714}$$

(d) Find the volume of the parallelopiped spanned by $\mathbf{u}, \mathbf{v}$ and $-\mathbf{j}$, where $\mathbf{j} = \langle 0, 1, 0 \rangle$.

Solution:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times -\mathbf{j}) &= \begin{vmatrix} 4 & 1 & 3 \\ -5 & 1 & 6 \\ 0 & -1 & 0 \end{vmatrix} \\ &= 4 \cdot 6 - 1 \cdot 0 + 3(5) \qquad \text{(expanding along the top row)} \\ &= 39 \end{aligned}$$

(e) Are $\mathbf{u}$ and $\langle -3, 4, -1 \rangle$ orthogonal? Why or why not?

Solution:

$$\begin{aligned} \mathbf{u} \cdot \langle -3, 4, -1 \rangle &= \langle 4, 1, 3 \rangle \cdot \langle -3, 4, -1 \rangle \\ &= -12 + 4 - 3 = -11 \neq 0 \end{aligned}$$

so <u>NO</u>, they are not orthogonal.

5. (10 points) §2.5: Let $P = (1, 1, -1)$, $Q = (-7, -6, 4)$, and $R = (-3, 2, 8)$.

(a) Describe the line $\mathcal{L}$ through P and Q in vector, parametric, and symmetric form.

Solution:

Vector form

$$\begin{aligned}\mathcal{L} &= \{P + t\,\overrightarrow{PQ} \mid t \in \mathbb{R}\}\\ &= \langle 1, 1, -1\rangle + t\,\langle -8, -7, 5\rangle\end{aligned}$$

Parametric form

$$\begin{aligned}x &= 1 - 8t,\\ y &= 1 - 7t,\\ z &= -1 + 5t\end{aligned}$$

Symmetric form

$$\frac{x-1}{-8} = \frac{y-1}{-7} = \frac{z+1}{5}.$$

(b) Find the distance from R to the line $\mathcal{L}$ through P and Q.

Solution: The distance between R and $\mathcal{L}$ is given by the length of the vector $\overrightarrow{RP} - \operatorname{proj}_{\overrightarrow{PQ}} \overrightarrow{RP}$, which is the portion of $\overrightarrow{RP}$ orthogonal to $\overrightarrow{PQ}$ (draw a picture to see this). We thus compute:

$$\begin{aligned}\operatorname{proj}_{\overrightarrow{PQ}} \overrightarrow{RP} &= \frac{\langle -4, 1, 9\rangle \bullet \langle -8, -7, 5\rangle}{64 + 49 + 25} \cdot \langle -8, -7, 5\rangle\\ &= \frac{70}{138}\,\langle -8, -7, 5\rangle\\ &= \frac{35}{69}\,\langle -8, -7, 5\rangle\\ \overrightarrow{RP} - \operatorname{proj}_{\overrightarrow{PQ}} \overrightarrow{RP} &= \langle -4, 1, 9\rangle - \frac{35}{69}\,\langle -8, -7, 5\rangle = \left\langle \frac{4}{69}, \frac{314}{69}, \frac{446}{69} \right\rangle\end{aligned}$$

which has length $\frac{1}{69}\sqrt{4^2 + 314^2 + 446^2} = \frac{1}{69}\sqrt{297528} = \boxed{\frac{14}{69}\sqrt{1518}}$

(c) Describe the plane Π through P, Q and R in vector, scalar, or general form (any form is acceptable).

Solution:

$$\overrightarrow{PQ} = \langle -8, -7, 5 \rangle, \overrightarrow{PR} = \langle -4, 1, 9 \rangle \text{ gives}$$

$$\begin{aligned} \mathbf{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -7 & 5 \\ -4 & 1 & 9 \end{vmatrix} \\ &= \langle -68, 52, -36 \rangle \end{aligned}$$

gives

$$\langle -68, 52, -36 \rangle \cdot \langle x-1, y-1, z+1 \rangle = 0$$

so Π is given by

$$-68x + 52y - 36z - 20 = 0.$$

(d) Classify the relationship between the line $\mathcal{L}$ through P, Q and the line $\mathcal{M}$ through R and $(-10, -5, 2)$. Are $\mathcal{L}$ and $\mathcal{M}$: equal, parallel but not equal, intersect but not equal, or skew?

Solution: Let

$$\begin{aligned} \mathcal{L} &= \mathbf{P} + t\mathbf{PQ} = \langle 1, 1, -1 \rangle + t \langle -8, -7, 5 \rangle \\ \mathcal{M} &= \mathbf{R} + t\mathbf{RA} = \langle -3, 2, 8 \rangle + u \langle -7, -7, -6 \rangle \end{aligned}$$

where $A := (-10, -5, 2)$. Since $\langle -8, -7, 5 \rangle \neq k \langle -7, -7, -6 \rangle$ for any $k \in \mathbb{R}$, the two lines are not parallel.

Next we check whether the lines have a point of intersection. Setting the component functions of the two lines equal to each other, we get the system of equations

$$\begin{cases} 1 - 8t = -3 - 7u \\ 1 - 7t = 2 - 7u \\ -1 + 5t = 8 - 6u \end{cases}$$

Solving the first two equations yields the solution $t = 5, u = \frac{36}{7}$. Plugging this solution into the third equation yields 24 for the LHS and $-\frac{160}{7}$ for the RHS, so the two lines do not intersect.

We have found that $\mathcal{L}$ and $\mathcal{M}$ are not parallel, and do not intersect. Therefore, $\mathcal{L}$ and $\mathcal{M}$ must be skew.

(e) Find the distance from $(2, -2, 2)$ to the plane Π.

Solution: Let $B = (2, -2, 2)$. We can take the line L' parallel to $\overrightarrow{PQ} \times \overrightarrow{PR}$ through B until it hits Π : That is, we look for a simultaneous solution to

$$x = 2 - 68t, \quad y = -2 + 52t, \quad z = 2 - 36t$$

and

$$-68x + 52y - 36z - 20 = 0$$

Plugging in gives

$$\begin{aligned} -68(2 - 68t) + 52(-2 + 52t) - 36(2 - 36t) - 20 = 0 \\ (68)^2 t + (52)^2 t + (36)^2 t = 20 + 136 + 104 + 72 \\ \implies t = \frac{332}{68^2 + 52^2 + 36^2} = \frac{83}{2156} \end{aligned}$$

Thus the point on Π closest to B is given by

$$Q' := (2, -2, 2) + \frac{332}{68^2 + 52^2 + 36^2} \cdot \langle -68, 52, -36 \rangle$$

and we can compute the distance between Π and B by computing $\|\overrightarrow{BQ'}\|$:

$$\begin{aligned} \left\| \overrightarrow{BQ'} \right\| &= \frac{332}{68^2 + 52^2 + 36^2} \cdot \sqrt{68^2 + 52^2 + 36^2} \\ &= \frac{332}{\sqrt{68^2 + 52^2 + 36^2}} = \boxed{\frac{83}{7\sqrt{11}}} \end{aligned}$$

6. (10 points) §2.6:

(a) Classify the following quadric surface and identify the axis of the surface:

$$49x^2 - 392x + 16y^2 - 32y - 784z + 2336 = 0$$

Solution: Completing the square:

$$\begin{aligned} 0 &= 49x^2 - 392x - 16y^2 + 32y - 784z + 2336 \\ 0 &= 49x^2 - 392x + 784 - 16y^2 + 32y - 16 - 784z + 1568 \\ 0 &= 49x^2 - 8 \cdot 49x + 16 \cdot 49 - 16y^2 + 32y - 16 - 16 \cdot 49z + 32 \cdot 49 \\ 16 \cdot 49z - 32 \cdot 49 &= 49\left(x^2 - 8x + 16\right) - 16\left(y^2 - 2y + 1\right) \\ 16 \cdot 49(z-2) &= 49(x-4)^2 - 16(y-1)^2 \\ (z-2) &= \frac{(x-4)^2}{16} - \frac{(y-1)^2}{49} \end{aligned}$$

gives a hyperbolic paraboloid.

(b) Give the trace in the $z = 5$ plane.

Solution: Setting $z = 5$, we have

$$\begin{aligned} 3 &= \frac{(x-4)^2}{16} - \frac{(y-1)^2}{49} \\ \implies 1 &= \frac{(x-4)^2}{48} - \frac{(y-1)^2}{147} \end{aligned}$$

(which is a hyperbola)

7. (10 points) §2.7:

(a) Identify the following surfaces in spherical coordinates, and sketch a graph of the surface:

i. $\phi = \frac{\pi}{2}$.

Solution: This is the xy-plane $\{z = 0\}$.

ii. $\rho = \cos\theta \sin\phi$. Hint: Multiply both sides by ρ.

Solution: Multiplying both sides by ρ gives the equation $\rho^2 = \rho\cos\theta\sin\phi$ which when converted to Cartesian coordinates becomes

$$\begin{aligned} x^2 + y^2 + z^2 &= x \\ x^2 - x + y^2 + z^2 &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + y^2 + z^2 &= 0 \\ \left(x - \frac{1}{2}\right)^2 + y^2 + z^2 &= \frac{1}{4} \end{aligned}$$

gives a sphere centered at $\left(\frac{1}{2}, 0, 0\right)$ with radius $\frac{1}{2}$.

(b) Convert the following points from rectangular to both cylindrical and spherical coordinates, respectively:

i. $(3, 1, 5)$.

Solution:

Cylindrical $(r, \theta, z) = (\sqrt{10}, \arctan\left(\frac{1}{3}\right), 5)$

Spherical $(\rho, \theta, \phi) = \left(\sqrt{35},\ \arctan\left(\frac{1}{3}\right),\ \arccos\left(\frac{5}{\sqrt{35}}\right)\right)$

ii. $(-2, 1, 7)$.

Solution:

Cylindrical $(r, \theta, z) = \left(\sqrt{5},\ \arccos\left(-\frac{2}{\sqrt{5}}\right),\ 7\right)$

Spherical $(\rho, \theta, \phi) = \left(\sqrt{54},\ \arccos\left(-\frac{2}{\sqrt{5}}\right),\ \arccos\left(\frac{7}{\sqrt{54}}\right)\right)$

Note: Because this point lies in Quadrant II when projected onto the xy-plane, more care is needed to express θ correctly. Instead of arccosine, we could have used $\theta = \arctan\left(-\frac{1}{2}\right) + \pi$ instead.

8. (10 points) §3.1: Let $\mathbf{r}(t) = \left\langle e^{-4t}, e^{\frac{1}{2t}}, \ln(2t) - 7 \right\rangle$.

(a) Find $\lim_{t\to 10} \mathbf{r}(t)$.

Solution: All component functions are continuous at $t = 10$. Thus we have

$$\lim_{t\to 10} \mathbf{r}(t) = \left\langle e^{-40}, e^{1/20}, \ln(20) - 7 \right\rangle.$$

(b) Is $\mathbf{r}(t)$ continuous at $t = 10$?

Solution: Yes, $\mathbf{r}(t)$ is continuous in each component on its domain, which includes $t = 10$.

(c) Are there any domain restrictions on $\mathbf{r}(t)$?

Solution: The second component function requires $2t \neq 0$ and the third component function requires that $2t > 0$. Hence $\text{Dom}(\mathbf{r}(t)) = \{t > 0\}$.

(d) Are there any values for t at which $\mathbf{r}(t)$ is not continuous?

Solution: No, $\mathbf{r}(t)$ is continuous on $\text{Dom}(\mathbf{r}(t))$.

(e) Let C be the space curve given by $\mathbf{r}(t) = \langle t, t^3, 20 \rangle$.
Describe the curve: What shape is it?

Solution: Cubic curve in the plane $\{z = 20\}$.

(f) C is contained in a unique plane in $\mathbb{R}^3$. Which plane is C contained in?

Solution: The plane $\{z = 20\}$. Note that the third coordinate is constant.

9. (10 points) §3.2:

(a) Using the limit definition of the derivative, find $\mathbf{r}'(t)$ for

$$\mathbf{r}(t) = \langle t^2, -4t \rangle.$$

Solution:

$$\begin{aligned}
\mathbf{r}'(t) := \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\
&= \lim_{h \to 0} \frac{\left\langle (t+h)^2, -4(t+h) \right\rangle - \left\langle t^2, -4t \right\rangle}{h} \\
&= \lim_{h \to 0} \frac{\left\langle 2th + h^2, -4h \right\rangle}{h} \\
&= \lim_{h \to 0} \frac{h}{h} \langle 2t + h, -4 \rangle = \boxed{\langle 2t, -4 \rangle}
\end{aligned}$$

(b) Using your formula from part (a), find $\mathbf{r}'(5)$.

Solution: $\mathbf{r}'(5) = \langle 10, -4 \rangle$.

(c) Let $\mathbf{u}(t) = \left\langle t, t^2, t^3 \right\rangle$ and $\mathbf{w}(t) = \left\langle 2t+3, \ln(t), 12 \right\rangle$. Find

$$\int \mathbf{u}(t) \times \mathbf{w}(t)\,\mathrm{d}t$$

Solution:

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 2t+3 & \ln(t) & 12 \end{vmatrix}$$

$$= \left\langle 12t^2 - \ln(t)\cdot t^3, -\left[12t - 2t^4 - 3t^3\right], t\ln(t) - (2t^3 + 3t^2) \right\rangle$$

$$= \left\langle 12t^2 - \ln(t)\cdot t^3,\ -12t + 2t^4 + 3t^3,\ t\ln(t) - 2t^3 - 3t^2 \right\rangle$$

so

$$\int \mathbf{u} \times \mathbf{w}\,\mathrm{d}t$$

$$= \int \left\langle 12t^2 - t^3\ln(t),\ -12t + 2t^4 + 3t^3,\ t\ln(t) - 2t^3 - 3t^2 \right\rangle \mathrm{d}t$$

$$= \left\langle \frac{12t^3}{3} + C_1 - \int t^3\ln(t)\,\mathrm{d}t,\ -6t^2 + \frac{2}{5}t^5 + \frac{3}{4}t^4 + C_2,\ \int t\ln(t)\,\mathrm{d}t - \frac{1}{2}t^4 - t^3 + C_3 \right\rangle$$

The remaining integrals are evaluated using integration-by-parts:

$$\int t^3\ln(t)\,\mathrm{d}t = \ln(t)\cdot\frac{t^4}{4} - \frac{1}{4}\int t^3\,\mathrm{d}t$$
$$= \frac{\ln(t)t^4}{4} - \frac{1}{16}t^4 + C$$

using $u = \ln(t)$, $\mathrm{d}v = t^3\,\mathrm{d}t$, and

$$\int t\ln(t)\,\mathrm{d}t = \ln(t)\cdot\frac{t^2}{2} - \frac{1}{2}\int t\,\mathrm{d}t$$
$$= \frac{\ln(t)t^2}{2} - \frac{1}{4}t^2 + C'$$

using $u = \ln(t)$, $\mathrm{d}v = t\,\mathrm{d}t$. Thus we have

$$\boxed{\left\langle 4t^3 - \frac{\ln(t)t^4}{4} + \frac{1}{16}t^4,\ -6t^2 + \frac{2}{5}t^5 + \frac{3}{4}t^4,\ \frac{\ln(t)t^2}{2} - \frac{1}{4}t^2 - \frac{1}{2}t^4 - t^3 \right\rangle + \langle C_1, C_2, C_3 \rangle}$$

(d) Find $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{w}(t)]$

Solution: In the previous part, we computed

$$\mathbf{u} \times \mathbf{w} = \left\langle 12t^2 - t^3 \ln(t),\, -12t + 2t^4 + 3t^3,\, t\ln(t) - 2t^3 - 3t^2 \right\rangle$$

Its derivative is thus

$$\boxed{= \left\langle 24t - t^2 - 3t^2 \ln(t),\, -12 + 8t^3 + 6t,\, 1 + \ln(t) - 6t^2 - 6t \right\rangle}$$

10. (10 points) §3.3: Let $\mathbf{r}(t) = \langle 3\cos(2t), 3\sin(2t), 12t\rangle$.

(a) Find the arc length of $\mathbf{r}(t)$ over $t \in [0, 4\pi]$.

Solution: We evaluate $\int_0^{4\pi} \|\mathbf{r}'(t)\| \,\mathrm{d}t$. First we compute

$$\begin{aligned}\mathbf{r}'(t) &= \big\langle -6\sin(2t), 6\cos(2t), 12\big\rangle,\\ \big\|\mathbf{r}'(t)\big\| &= \sqrt{36\left(\sin^2(2t) + \cos^2(2t)\right) + 144}\\ &= \sqrt{180} = 6\sqrt{5}\end{aligned}$$

Thus

$$\int_0^{4\pi} \|\mathbf{r}'(t)\| \,\mathrm{d}t = \int_0^{4\pi} 6\sqrt{5}\,\mathrm{d}t = \boxed{24\pi\sqrt{5}}$$

(b) Give an arc length parametrization of $\mathbf{r}(t)$ for $t > 0$.

Solution: The arc length function is

$$s(t) = \int_0^t \|\mathbf{r}'(u)\| \,\mathrm{d}u = 6\sqrt{5}t$$

by the computation in the previous part. The inverse of the arc length function is thus

$$t(s) = \frac{s}{6\sqrt{5}}.$$

Hence

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{r}(t(s))\\ &= \left\langle 3\cos\left(2\frac{s}{6\sqrt{5}}\right), 3\sin\left(2\frac{s}{6\sqrt{5}}\right), 12\frac{s}{6\sqrt{5}}\right\rangle\\ &= \boxed{\left\langle 3\cos\left(\frac{1}{3\sqrt{5}}\right),\ 3\sin\left(\frac{s}{3\sqrt{5}}\right),\ \frac{2s}{\sqrt{5}}\right\rangle}\end{aligned}$$

(c) Find the principal unit normal vector $\mathbf{N}(t)$ at $t = 4\pi$.

Solution: A series of calculations are required. We use the definitions that

$$\mathbf{T}(t) := \frac{\mathbf{r}'}{\|\mathbf{r}'\|} \qquad \text{and} \qquad \mathbf{N}(t) := \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

Then

$$\begin{aligned}
\mathbf{r}(t) &= \big\langle 3\cos(2t),\ 3\sin(2t),\ 12t \big\rangle \\
\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\big\|\mathbf{r}'(t)\big\|} &= \frac{1}{6\sqrt{5}} \big\langle -6\sin(2t), 6\cos(2t), 12 \big\rangle \\
\mathbf{T}'(t) &= \frac{1}{6\sqrt{5}} \big\langle -12\cos(2t), -12\sin(2t), 0 \big\rangle \\
\|\mathbf{T}'\| &= \frac{1}{6\sqrt{5}} \cdot 12 = \frac{2}{\sqrt{5}} \\
\mathbf{N}(t) := \frac{\mathbf{T}'}{\|\mathbf{T}'\|} &= \frac{1}{12} \big\langle -12\cos(2t), -12\sin(2t), 0 \big\rangle \\
&= \big\langle -\cos(2t), -\sin(2t), 0 \big\rangle
\end{aligned}$$

Thus $\mathbf{N}(4\pi) = \boxed{\langle -1, 0, 0 \rangle}$