Name:____________________________________

1. Find all solutions to $\frac{dy}{dx} = \frac{3x-5y}{5x+2y}$.

$$(5x+2y)\,dy = (3x-5y)\,dx$$

$$(5y-3x)\,dx + (5x+2y)\,dy = 0$$

$$\frac{\partial(5y-3x)}{\partial y} = 5 = \frac{\partial(5x+2y)}{\partial x} \qquad \underline{\text{EXACT}}$$

$$\frac{\partial F}{\partial x} = 5y-3x \longrightarrow F = \int 5y-3x\,\partial x = 5xy - \tfrac{3}{2}x^2 + C(y)$$

$$\frac{\partial F}{\partial y} = 5x+2y \longrightarrow F = \int 5x+2y\,\partial y = 5xy + y^2 + \tilde{C}(x)$$

$$F = \boxed{5xy + y^2 - \tfrac{3}{2}x^2 = K}$$

Name:____________________________________

2. Find $y(\pi/2)$ where $y(x)$ is the solution to

$$\frac{dy}{dx} = \cos(x)y, \quad y(0) = 1$$

SEPARABLE

$$\frac{dy}{y} = \cos(x)\,dx$$

$$\int \frac{dy}{y} = \int \cos(x)\,dx$$

$$\ln|y| = \sin(x) + C$$

$$y = ke^{\sin(x)}$$

Since $y(0)=1$ $\quad 1 = ke^{\sin(0)} = ke^0 = k$

So $\quad y = e^{\sin(x)}$

Hence $\quad y(\pi/2) = e^{\sin(\pi/2)} = e^1 = \boxed{e}$

Name:__________________________________

3. Solve the initial value problem, $\frac{dy}{dx} - \sin(x)\,y = \cos(x), \quad y(0) = 2$. Your answer will involve an integral.

LINEAR

$$\mu = e^{-\int \sin(x)\,dx} = e^{\cos(x) + \cancel{C}^{\,0}}$$

$$\underbrace{e^{\cos x}\frac{dy}{dx} - e^{\cos(x)}\sin(x)\,y}_{\frac{d}{dx}\left(e^{\cos(x)}y\right)} = e^{\cos(x)}\cos(x)$$

$$\frac{d}{dx}\left(e^{\cos(x)}y\right) = e^{\cos(x)}\cos(x) \quad \text{(can't simplify)}$$

$$e^{\cos(x)}y = \int_0^x e^{\cos(t)}\cos(t)\,dt + C$$

$$y = e^{-\cos(x)}\int_0^x e^{\cos(t)}\cos(t)\,dt + Ce^{-\cos(x)}$$

$$y(0) = 2 \rightarrow \quad 2 = e^{-\cos(0)}\underbrace{\int_0^0 e^{\cos(t)}\cos(t)\,dt}_{=0 \text{ since } \int_0^0} + Ce^{-\cos(0)}$$

$$2 = Ce^{-1} \rightarrow C = 2e$$

$$\boxed{y(x) = e^{-\cos(x)}\int_0^x e^{\cos(t)}\cos(t)\,dt + \underbrace{2e\,e^{-\cos(x)}}_{\text{OR } 2e^{1-\cos(x)}}}$$

Name:______________________________________

4. Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{y^2 - 6x^2}{2xy - 5x^2}, \qquad y(1) = 3$$

$$(6x^2 - y^2)\,dx + (2xy - 5x^2)\,dy = 0$$

$$\frac{\partial}{\partial y}(6x^2 - y^2) = -2y \qquad \frac{\partial}{\partial x}(2xy - 5x^2) = 2y - 10x \qquad \text{Rats - Not exact}$$

HOMOGENEOUS (all terms degree 2)

$$\frac{dy}{dx} = \frac{(y/x)^2 - 6}{2(y/x) - 5}$$

$$y = xv \qquad v + x\frac{dv}{dx} = \frac{v^2 - 6}{2v - 5}$$

$$x\frac{dv}{dx} = \frac{v^2 - 6}{2v - 5} - v = \frac{v^2 - 6 - 2v^2 + 5v}{2v - 5}$$

$$x\frac{dv}{dx} = -\frac{v^2 - 5v + 6}{2v - 5}$$

$$-\underbrace{\int \frac{2v - 5\,dv}{v^2 - 5v + 6}}_{\text{Partial Fractions}} = \int \frac{dx}{x} = \log|x| + C$$

$$\frac{2v - 5}{v^2 - 5v + 6} = \frac{A}{v - 3} + \frac{B}{v - 2}$$

$$2v - 5 = A(v - 2) + B(v - 3)$$

$v = 2 \qquad -1 = -B \to B = 1$

$v = 3 \qquad 1 = A \to A = 1$

Continued on next page

(that's why I always include a blank page at the end of the test)

Name:____________________________________

So $$-\int \frac{2v-5}{v^2-5v+6}\,dv = -\int \frac{dv}{v-2} - \int \frac{dv}{v-3}$$

$$= -\log|v-2| - \log|v-3|$$

$$-\log|v-2| - \log|v-3| = \log|x| + C$$

$$\frac{1}{(v-2)(v-3)} = kx$$

$$\frac{1}{(y/x-2)(y/x-3)} = kx$$

Multiply top and bottom by x^2

$$\frac{x^2}{(y-2x)(y-3x)} = kx$$

$$x^2 = kx(y-2x)(y-3x)$$

General solution

$$\underline{x = k(y-2x)(y-3x)}$$

Now since $y(1)=3$

$$1 = k(3-2)(3-3)$$

$$1 = 0$$

?

<u>Remember singular solutions</u>

You divided by $(v-2)(v-3)$

$v=2$, $v=3$ are both solutions

$y/x = 2$, $y/x = 3$

$y = 2x$ $\quad$ $\underline{y=3x}$

satisfies $y(1)=3$

So solution is $y=3x$

If you remembered the singular solution when solving the separable equation (as you should) and spotted it worked, you could jump straight here.

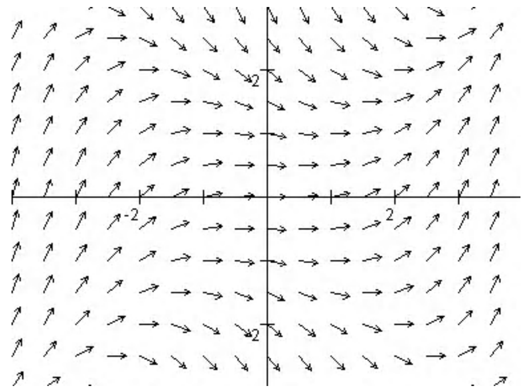
Name:________________________________

5. Match the equations on the left with the slope fields on the right.

$y' = 0.3(x^2 - x - 2)$ 2

fct. of x alone
slopes constant on
vertical lines

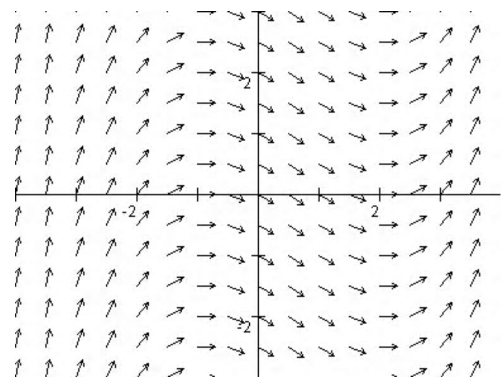
(1)



$y' = 0.3(y^2 - y - 2)$ 4

fct. of y alone
slopes constant on
horizontal lines
y=0 , y' = -0.6 <0
So ↘ along x-axis

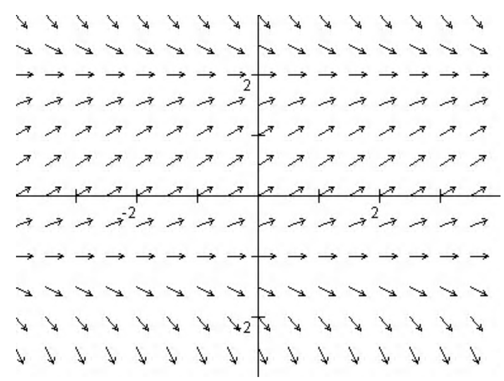
(2)



$y' = -0.3(y^2 - y - 2)$ 3

fct. of y alone
slopes constant on
horizontal lines
y=0 , y' = 0.6 >0
So ↗ along x-axis

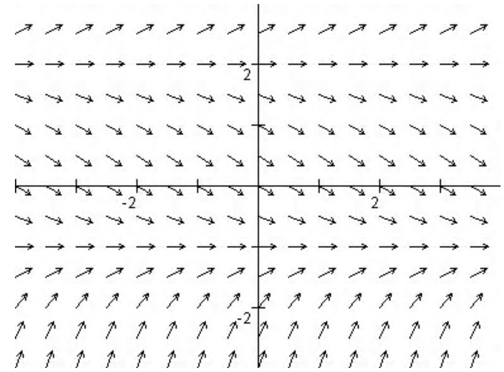
(3)



$y' = 0.2(x^2 - y^2)$ 1

fct. of x and y
slopes vary on both horizontal
and vertical lines

(4)



Name:______________________________________

6. Using the improved Euler method with step size $h = 1$, approximate $y(2)$ if $\frac{dy}{dx} = 2x - y$, $y(0) = -1$.

$h=1$

x	left y'	$\tilde{y}$	right y'	y
0				-1
$0+1=1$	$2\cdot 0-(-1)$ $=1$	$-1+1\cdot 1$ $=0$	$2\cdot 1-0$ $=2$	$-1+1\cdot\frac{1+2}{2}$ $=\frac{1}{2}$
$1+1=2$	$2\cdot 1-\frac{1}{2}$ $=\frac{3}{2}$	$\frac{1}{2}+1\cdot\frac{3}{2}$ $=2$	$2\cdot 2-2$ $=2$	$\frac{1}{2}+1\cdot\frac{3/2+2}{2}$ $=\frac{9}{4}=2.25$

$$\boxed{\text{So } y(2) \approx 2.25}$$

Name:____________________________________

7. Find a differential equation whose general solution is the set of all ellipses of the form $\left(\frac{x}{2}\right)^2 + y^2 = a^2$. Note that a is an arbitrary constant and should *not* appear in the differential equation.

$$\frac{x^2}{4} + y^2 = a^2$$

Differentiate both sides

$$\frac{2x}{4} + 2y\frac{dy}{dx} = 0$$

any of these forms are acceptable

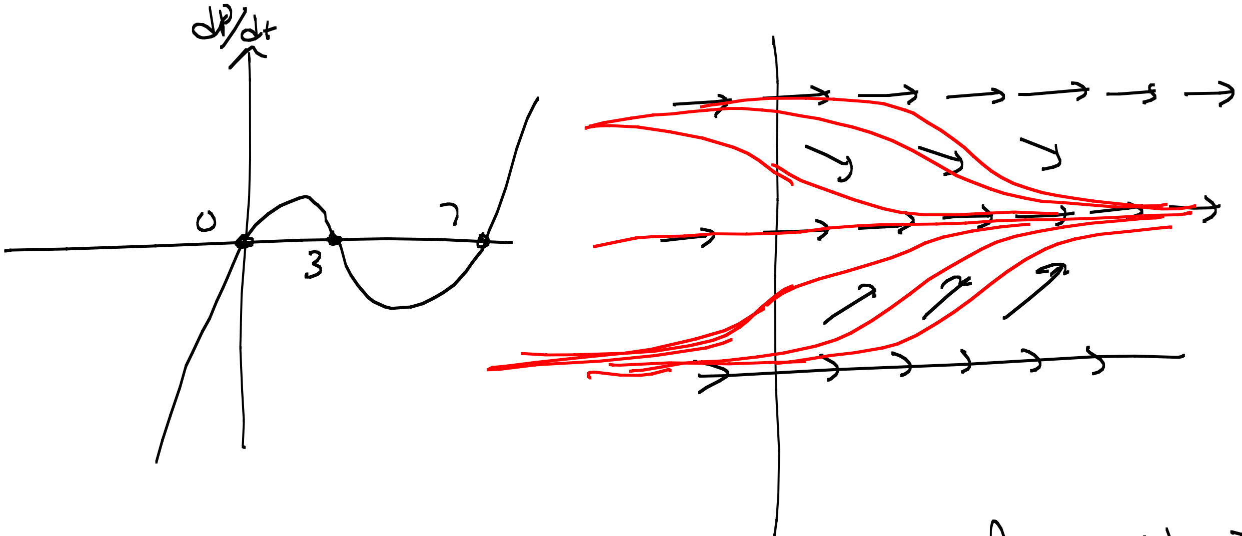
$$2y\frac{dy}{dx} = -\frac{x}{2}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{4y}}$$

Name:______________________________________

8. Suppose $P(t)$ is a solution of $\frac{dP}{dt} = P^3 - 10P^2 + 21P$ with $\lim_{t\to\infty} P(t) = 3$. What can you say about the value of $\lim_{t\to-\infty} P(t)$? Justify your answer, using complete sentences.

$P^3 - 10P^2 + 21P = P(P-3)(P-7)$



Looking at the slope field, we see $\lim_{t\to\infty} P(t) = 3$ provided $0 < P(0) < 7$. Now looking as t tends to $-\infty$ we see that if $0 < P(0) < 3$, $\lim_{t\to-\infty} P(t) = 0$ while if $3 < P(0) < 7$, $\lim_{t\to-\infty} P(t) = 7$. But these two cases miss $P(0) = 3$ and for that one value $\lim_{t\to-\infty} P(t) = 3$.

If you found 0 and 7 but missed 3 you lost 1 point. You got half credit for finding 0 or 7 but missing the other value.