Name:____________________________________

**1.** Solve the initial value problem $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+10y=0$, $y(0) = 3$, $y'(0) = 9$.

$$(D^2+7D+10)y = 0$$

$$(D+5)(D+2)y = 0$$

roots $-5, -2$

General Solution

$$y = C_1e^{-5x}+C_2e^{-2x}$$

$$y' = -5C_1e^{-5x}-2C_2e^{-2x}$$

Initial Conditions

$$y(0) = C_1 + C_2 = 3$$

$\times 2$

$$y'(0) = -5C_1 -2C_2 = 9$$

$$2C_1 + 2C_2 = 6$$

$$-3C_1 = 15$$

$$C_1 = -5 \Rightarrow -5 + C_2 = 3$$

$$C_2 = 8$$

$$\boxed{y = -5e^{-5x}+8e^{-2x}}$$

Name:______________________________________

**2.** Find the general solution to $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+17y=\exp(-x)$.

Homo. Soln. $(D^2+2D+17)y=0$

roots $\dfrac{-2\pm\sqrt{4-4\cdot 17}}{2}=\dfrac{-2\pm\sqrt{-64}}{2}$

$=-1\pm 4i$

$y_h = C_1 e^{-x}\cos(4x)+C_2 e^{-x}\sin(4x)$

Part. Soln. Guess $y_p = Ae^{-x}$

$y_p' = -Ae^{-x}$

$y_p'' = Ae^{-x}$

$y_p''+2y_p'+17y_p = (A-2A+17A)e^{-x} \overset{\text{Set}}{=} e^{-x}$

$16A = 1$

$A = 1/16$

$y_p = \frac{1}{16}e^{-x}$

$$\boxed{y=\frac{1}{16}e^{-x}+C_1e^{-x}\cos(4x)+C_2e^{-x}\sin(4x)}$$

Name:____________________________________

**3.** Solve the initial value problem $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} \cancel{+} 8y = 5\exp(3x).$
$y(0) = 4, \quad y'(0) = 5$

Homo. Soln.

$$D^2 - 2D - 8y = 0$$

$$(D-4)(D+2)$$

roots $4, -2$

$$y_h = C_1 e^{4x} + C_2 e^{-2x}$$

Part Soln.

$$y_p = Ae^{3x}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y_p'' - 2y_p' - 8y_p = (9-6-8)Ae^{3x} = -5Ae^{3x}$$

$$A = -1$$

$$y_p = -e^{3x}$$

General Soln.

$$y = -e^{3x} + C_1 e^{4x} + C_2 e^{-2x}$$

$$y' = -3e^{3x} + 4C_1 e^{4x} - 2C_2 e^{-2x}$$

Initial Conditions

$$y(0) = -1 + C_1 + C_2 = 4 \quad \times 2$$

$$y'(0) = -3 + 4C_1 - 2C_2 = 5$$

$$-2 + 2C_1 + 2C_2 = 8$$

$$-5 + 6C_1 = 13$$

$$6C_1 = 18$$

$$C_1 = 3 \Rightarrow -1 + 3 + C_2 = 4$$

$$C_2 = 2$$

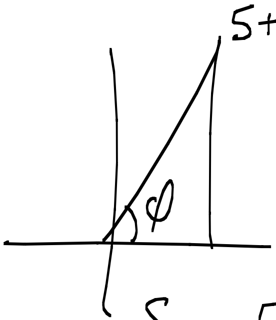
$$\boxed{y = -e^{3x} + 3e^{4x} + 2e^{-2x}}$$

Name:____________________________________

**4.** Rewrite $5\cos(9x)-12\sin(9x)$ in the form $A\cos(9x+\phi)$.

$$5\cos(9x)-12\sin(9x)$$

$$=\text{Re}\left[(5+12i)(\cos(9x)+i\sin(9x))\right]$$




$$|5+12i| = \sqrt{5^2+12^2} = 13$$

$$\tan\phi = \frac{12}{5} \quad \text{atan}\left(\frac{12}{5}\right) = 1.176\ldots$$

$1^{st}$ quadrant ✓

So $5+12i$

$$= 13e^{i(1.176\ldots)}$$

$$= \text{Re}\left[13e^{i(1.176\ldots)}e^{i9x}\right]$$

$$= \text{Re}\left[13e^{i(9x+1.176\ldots)}\right]$$

$$= \boxed{13\cos(9x+1.176\ldots)}$$

Name:________________________________________

**5.** Using Euler's method with step size $h = 0.1$, approximate $y(0.2)$ if $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0, \quad y(0) = 2, \quad y'(0) = 1.$

$$v = \frac{dy}{dx}$$

$$\frac{dv}{dx} = \frac{d^2y}{dx^2} = -5\frac{dy}{dx} - 6y = -5v - 6y$$

| $x$ | $\frac{dy}{dx}$ | $\frac{dv}{dx}$ | $y$ | $v$ |
|---|---|---|---|---|
| $0$ | | | $2$ | $1$ |
| $0+0.1$ $=0.1$ | $=1$ | $=-5\cdot1-6\cdot2$ $=-17$ | $2+.1(1)$ $=2.1$ | $1+.1(-17)$ $=-0.7$ |
| $0.1+0.1$ $=0.2$ | $=-0.7$ | $=-5(-0.7)-6(2.1)$ $=-9.1$ | $=2.1+0.1(-0.7)$ $=2.03$ | $-0.7+0.1(-9.1)$ $-1.61$ |

$$\boxed{y(0.2) \approx 2.03}$$

Don't actually need to compute final $\frac{dv}{dx}$ and $v$ since you are asked about $y$

Name:____________________________________

**6.** Match the following differential equations with the graphs of one of their solutions.

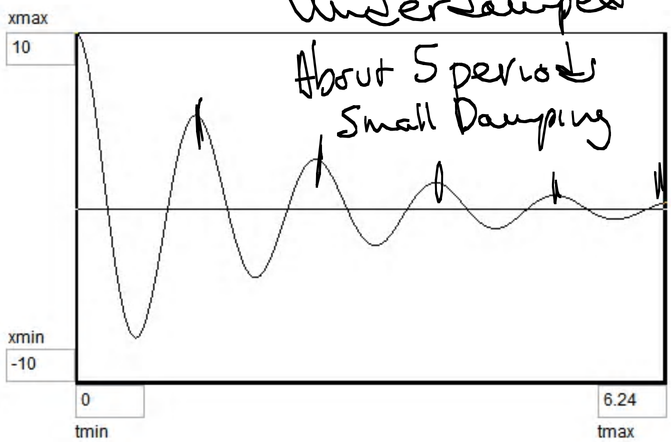
(a) $x''+x'+16x=0, \quad x(0)=10, \; x'(0)=0$

(b) $x''+x'+25x=0, \quad x(0)=10, \; x'(0)=0$

(c) $x''+10x'+16x=0, \quad x(0)=10, \; x'(0)=0$

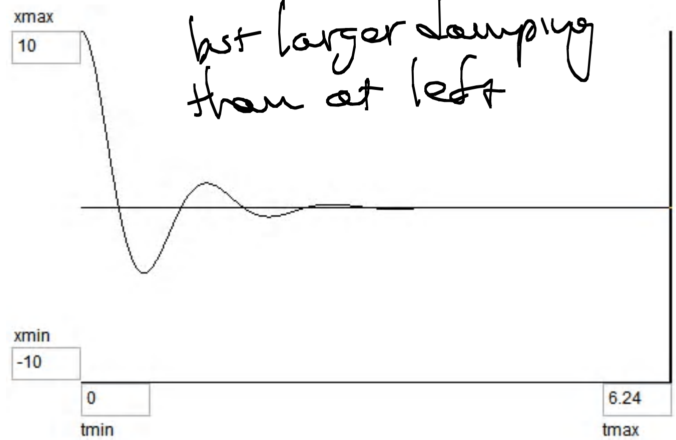
(d) $x''+3x'+25x=0, \quad x(0)=10, \; x'(0)=0$

Underdamped
About 5 periods
Small Damping




Equation B

Underdamped
but larger damping
than at left




Equation D

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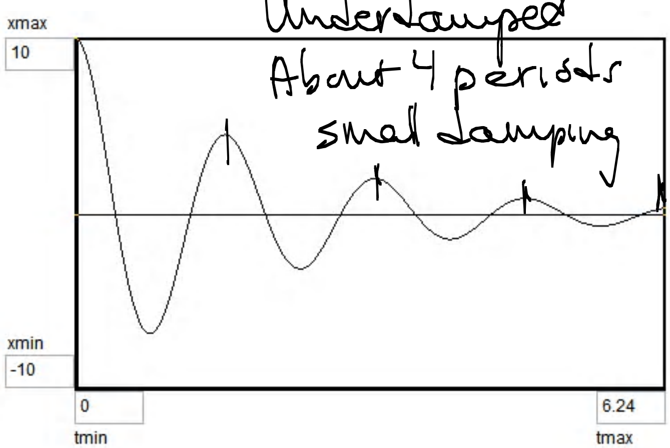
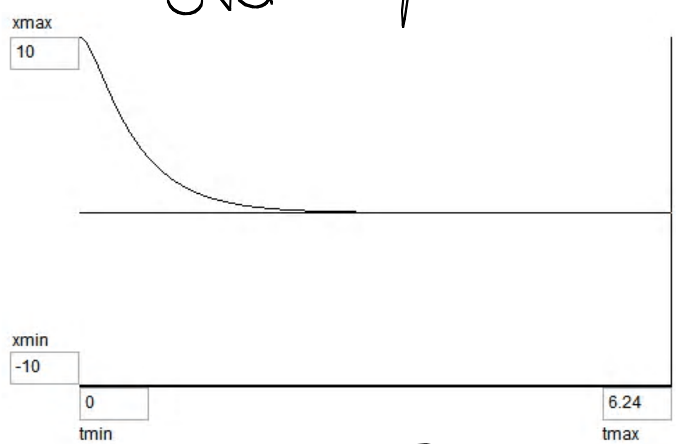
Underdamped
About 4 periods
small damping




Equation A

Overdamped




Equation C

Name:____________________________________

**7.** A spring-mass system has mass $m$, spring constant $k$, and hence natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$. The damping constant can take any value. Show that the smallest half-life you can get without the spring becoming overdamped is $\frac{\ln(2)}{\omega_0}$.

The larger the damping the smaller the half-life (as long as not overdamped)

Largest damping you can have before overdamping is critically damped at

$$c^2 = 4km$$

$$c = \sqrt{4km}$$

Then amplitude $= A e^{-\frac{ct}{2m}}$

$$= A e^{-\frac{\sqrt{4km}}{2m} t} = A e^{-\sqrt{\frac{k}{m}} t}$$

At time 0, amplitude $= A e^0 = A$

After 1 half-life Amplitude $= A/2$

So we want $t$ so that

$$A e^{-\sqrt{\frac{k}{m}} t} = \frac{1}{2} A \qquad \left(\text{remember } \omega_0 = \sqrt{\frac{k}{m}}\right)$$

$$e^{-\omega_0 t} = \frac{1}{2}$$

$$-\omega_0 t = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\boxed{t = \frac{\ln(2)}{\omega_0}}$$

Name:__________________________________

**8.** Suppose $L$ is a second-order linear differential equation, not necessarily monic or with constant coefficients, such that

$L(\exp(2x)) = 24\exp(2x)$
$L(\exp(-x)) = 0$
$L(\exp(-2x)) = 0$

Find the general solution to $Ly = 48\exp(2x)$.

Hint: while it is possible to solve this problem by deducing what $L$ is, there is a quicker way to the answer where you don't have to find $L$ explicitly.

By linearity, if $L(\exp(2x)) = 24\exp(2x)$

then $L(2\exp(2x))$
$= 2L(\exp(2x)) = 2(24\exp(2x))$
$= 48\exp(2x)$

So $2\exp(2x)$ is a particular solution

and $L(c_1 \exp(-x) + c_2 \exp(-2x))$
$= c_1 L(\exp(-x)) + c_2 L(\exp(-2x))$
$= c_1 \cdot 0 + c_2 \cdot 0 = 0$

So $c_1 \exp(-x) + c_2 \exp(-2x)$ is the general homogeneous solution (since 2 arbitrary constants and $e^{-x}, e^{-2x}$ linearly independent)

So the general solution is $\boxed{y = 2\exp(2x) + c_1 \exp(-x) + c_2 \exp(-2x)}$

Name:______________________________________

Alternate approach to #8

$L(e^{-x}) = 0$ so factor $D+1$

$L(e^{-2x}) = 0$ so factor $D+2$

Try $L = (D+1)(D+2) = D^2 + 3D + 2$

then $L(e^{2x}) = 4e^{2x} + 6e^{2x} + 2e^{2x}$

$= 12e^{2x}$ OOPS we wanted $24e^{2x}$

Multiply by 2 to fix this

$$\underline{L = 2D^2 + 6D + 4}$$

So our equation is

$$2\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 4y = 48e^{2x}$$

Now solve this as usual to get

$$\underline{y = 2e^{2x} + C_1 e^{-x} + C_2 e^{-2x}}$$