Name:___________________________________

1. Note: you may use the table of Laplace transforms which is attached at the end of the test.

a) Find the Laplace transform of $f(t) = t\cosh(t)$.

$$t\cosh(t) = t\left(\frac{e^t + e^{-t}}{2}\right) = \frac{1}{2}te^t + \frac{1}{2}te^{-t}$$

$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$ so

$$\mathcal{L}\{t\cosh t\} = \frac{1}{2}\mathcal{L}\{te^t\} + \frac{1}{2}\mathcal{L}\{te^{-t}\}$$

$$= \frac{1}{2}\frac{1}{(s-1)^2} + \frac{1}{2}\frac{1}{(s+1)^2} = \frac{\frac{1}{2}(s+1)^2 + \frac{1}{2}(s-1)^2}{(s-1)^2(s+1)^2}$$

$$= \frac{\frac{1}{2}(s^2+2s+1) + \frac{1}{2}(s^2-2s+1)}{[(s-1)(s+1)]^2} = \boxed{\frac{s^2+1}{(s^2-1)^2}}$$

b) Find the inverse Laplace transform of $F(s) = \dfrac{2s+9}{s^2+8s+17}$.

$$\frac{2s+9}{s^2+8s+17} = \frac{2s+9}{s^2+8s+16+1} = \frac{2s+9}{(s+4)^2+1^2}$$

(Partial Fractions)

$$\frac{2s+9}{(s+4)^2+1^2} = A\frac{s+4}{(s+4)^2+1^2} + B\frac{1}{(s+4)^2+1^2}$$

$$2s+9 = A(s+4) + B = As + 4A + B$$

s: $2 = A$ $\Rightarrow$ $A = 2$

Const: $9 = 4A + B$ $\Rightarrow$ $B = 1$

$$\boxed{f(t) = 2e^{-4t}\cos(t) + e^{-4t}\sin(t)}$$

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2. Solve the initial value problem

$$y''+6y'+8y=\delta(t), \quad y(0)=0, \quad y'(0)=0.$$

Since $\delta(t)$ and constant coefficients initial value problem, use Laplace transforms.

Let $Y = \mathcal{L}\{y\}$

$$\mathcal{L}\{y''+6y'+8y\} = \mathcal{L}\{\delta(t)\}$$

$$s^2 Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 6(sY - \cancel{y(0)}^{0}) + 8Y = 1$$

$$(s^2+6s+8)Y = 1$$

$$Y = \frac{1}{s^2+6s+8} = \frac{1}{(s+4)(s+2)} = A\frac{1}{s+4} + B\frac{1}{s+2}$$

$$1 = A(s+2) + B(s+4)$$

$$(s=-4) \quad 1 = -2A \to A = -\tfrac{1}{2}$$

$$(s=-2) \quad 1 = 2B \to B = \tfrac{1}{2}$$

$$\boxed{y(t) = \left(\tfrac{1}{2}e^{-2t} - \tfrac{1}{2}e^{-4t}\right)u(t)}$$

Alternatively

$$\frac{1}{s^2+6s+8} = \frac{1}{s^2+6s+9-1} = \frac{1}{(s+3)^2-1^2}$$

$$\boxed{y(t) = e^{-3t}\sinh(t)\, u(t)}$$

Name:______________________________

Variable Coefficient so use a power series

3. Solve the initial value problem,

$$y''+2xy'+4y=0, \quad y(0)=1, \; y'(0)=2.$$

Step 1

$$y=\sum_{n=0}^{\infty} a_n x^n \rightarrow 4y=\sum_{n=0}^{\infty} 4a_n x^n \quad ①$$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \rightarrow 2xy=\sum_{n=1}^{\infty} 2n a_n x^n \quad ②$$

$$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad ③$$

Step 2

Terms ① and ② are already in the correct form. For ③ we make the change

$j=n-2$

So $n=j+2$

$n=2 \rightarrow j=0$

$$y''=\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j$$

Step 3

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} 4a_m x^m = 0$$

Step 4

$$\underbrace{(2a_2+4a_0)}_{(m=0)} + \sum_{m=1}^{\infty}\underbrace{\left((m+2)(m+1)a_{m+2}+2ma_m+4a_m\right)}_{(m\geq 1)}x^m = 0$$

Step 5

$(m=0)$ $\quad 2a_2+4a_0=0 \rightarrow a_2=-2a_0$

$(m\geq 1)$ $\quad (m+2)(m+1)a_{m+2}+2ma_m+4a_m=0$

$$(m+2)(m+1)a_{m+2}=-(2m+4)a_m$$

$$a_{m+2}=-\frac{(2m+4)a_m}{(m+2)(m+1)}=-\frac{2\cancel{(m+2)}a_m}{\cancel{(m+2)}(m+1)}$$

$$a_{m+2}=-\frac{2a_m}{m+1} \quad \text{(the recurrence relation)}$$

Step 6

$a_0=y(0)=1$

$a_1=y'(0)=2$

$a_2=-2a_0=-2$

$(m=1)$ $\quad a_3=-\frac{2a_1}{2}=-2$

$(m=2)$ $\quad a_4=-\frac{2a_2}{3}=-\frac{2}{3}(-2)=\frac{4}{3}$

$$\boxed{y(x)=1+2x-2x^2-2x^3+\frac{4}{3}x^4+\cdots}$$

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4. Find the general solution for the **Euler equation**

$$x^2 y'' + 6xy' + 6y = 0$$

Linear Homogeneous so just need to find two linearly independent solutions

Guess $y = x^r$ (since it is an Euler equation)

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

$$x^2 y'' + 6xy' + 6y = r(r-1)x^r + 6r x^r + 6x^r$$

$$= (r(r-1) + 6r + 6) x^r = 0 \quad \text{for all } x$$

So $r(r-1) + 6r + 6 = 0$

$$r^2 - r + 6r + 6 = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0$$

$$r = -2, \; r = -3$$

$$\boxed{y = c_1 x^{-2} + c_2 x^{-3}}$$

Name:___________________________________

5. Match the systems on the left with the graphs on the right.

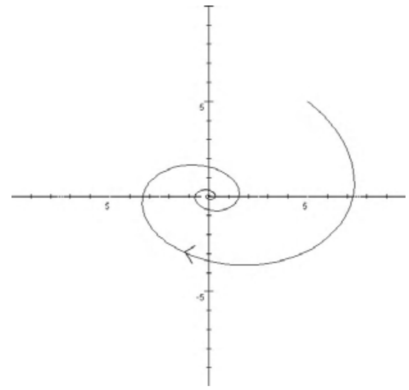
$$\frac{dx}{dt} = 0.2x + 2y \qquad \underline{C}$$
$$\frac{dy}{dt} = -x + 0.5y$$

It is a spiral out, but since there is only one outward graph we don't need to check this

$t = 2.1 > 0$ outward

trace $= 0.7 > 0$

(A)



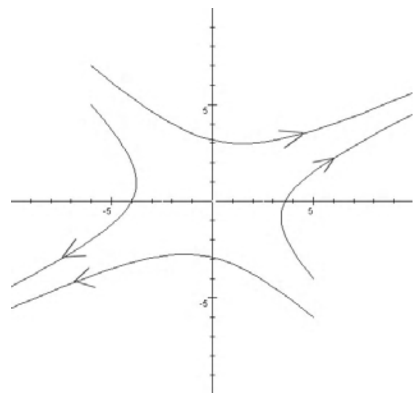
$$\frac{dx}{dt} = -0.2x + 2y \qquad \underline{A}$$
$$\frac{dy}{dt} = -x - 0.5y$$

It is a spiral in but since there is only one inward graph we don't need to check this

det $= 2.1 > 0$ inward

tr $= -0.7 < 0$

(B)

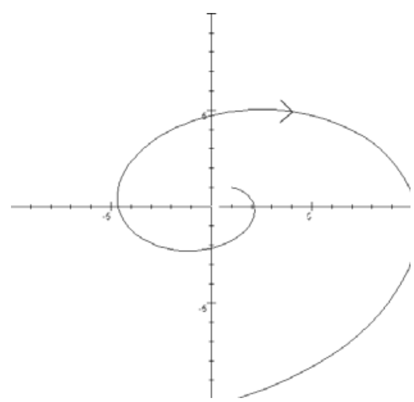


$$\frac{dx}{dt} = 0.5x + 2y \qquad \underline{D}$$
$$\frac{dy}{dt} = -x - 0.5y$$

det $= 1.75 > 0$ elliptical orbits

tr $= 0$

(C)

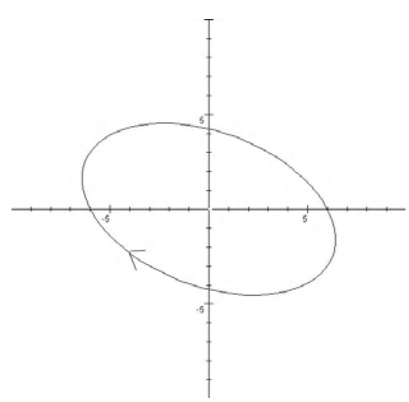


$$\frac{dx}{dt} = 0.5x + 2y \qquad \underline{B}$$
$$\frac{dy}{dt} = x - 0.5y$$

det $= -2.25 < 0$

Saddle

(D)



Name:___________________________________

6. Solve the initial value problem (your solution will involve an integral)

$$y''+4y=f(t), \quad y(0) = 0, \quad y'(0) = 0$$

Constant Coefficient initial value problem with unknown right hand side. While you can use variation of parameters, a convolution integral is easier

let $\bar{Y} = \mathcal{L}\{y\}$

$F = \mathcal{L}\{f\}$

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{f\}$$

$$s^2\bar{Y} - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 4\bar{Y} = F$$

$$(s^2+4)\bar{Y} = F$$

$$\bar{Y} = \frac{1}{s^2+4}F$$

$$\frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+2^2}$$

$$= \mathcal{L}\{\tfrac{1}{2}\sin(2t)\}$$

$$\bar{Y} = \mathcal{L}\{\tfrac{1}{2}\sin(2t)\}\mathcal{L}\{f(t)\}$$

$$\boxed{y(t) = \int_0^t \frac{1}{2}\sin(2u)\, f(t-u)\, du}$$

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7. Find the “a priori” lower bound for the radius of convergence of the series solution about $x_0 = 0$ to the equation

$$(x^2+2x+10)(x^2+6x+8)y''+2xy'-4y=0.$$

Singular points where

$$(x^2+2x+10)(x^2+6x+8)=0$$

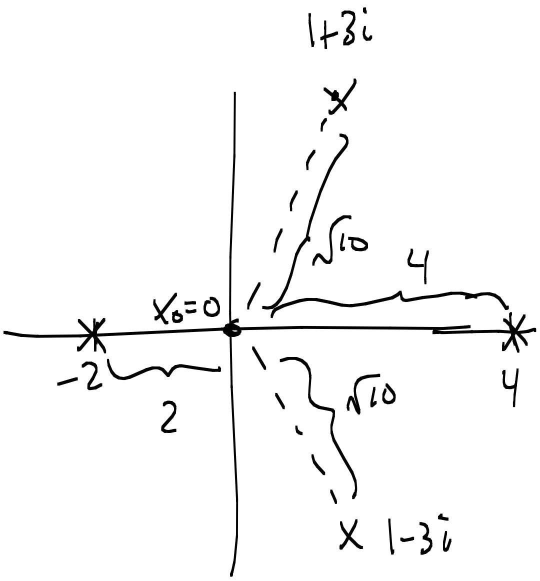
$x^2+2x+10=0$ OR $x^2+6x+8=0$

$\frac{-2\pm\sqrt{4-40}}{2}$ $\qquad$ $(x+4)(x+2)=0$

$\frac{-2\pm\sqrt{-36}}{2}$ $\qquad$ $x+4=0$ OR $x+2=0$

$\frac{-2\pm 6i}{2}$ $\qquad$ $x=-4$ $\quad$ $x=-2$

$-1\pm 3i$



Closest singular point is 2 away from $x_0=2$

So Radius of Convergence is at least 2

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8. Suppose x and y satisfy the system $\begin{aligned}\frac{dx}{dt} &= x+3y, \quad x(0)=a \\ \frac{dy}{dt} &= x-y, \quad y(0)=b\end{aligned}$. Find values of a and b so that X, the Laplace transform of x, has only one pole. Show that Y, the Laplace transform of y, will also only have one pole for the initial values you find.

Take Laplace transform of both sides $\mathcal{L}\{x\} = X \quad \mathcal{L}\{y\} = Y$

$$sX - \cancel{x(0)} \to a = X + 3Y$$

$$sY - \cancel{y(0)} \to b = X - Y$$

To solve for X, multiply the top equation by $(s+1)$ and the bottom equation by 3 and add

$$(s-1)X - 3Y = a \longrightarrow (s+1)(s-1)X - 3(s+1)Y = a(s+1)$$

$$-X + (s+1)Y = b \longrightarrow -3X + 3(s+1)Y = 3b$$

$$[(s+1)(s-1) - 3]X = as + a + 3b$$

$$(s^2-4)X = as + (a+3b)$$

$$X = \frac{as + (a+3b)}{s^2-4}$$

$$\boxed{X = \frac{as + (a+3b)}{(s+2)(s-1)}}$$

To solve for Y, multiply the bottom equation by $s-1$ and add

$$(s-1)X - 3Y = a$$

$$-(s-1)X + (s-1)(s+1)Y = b(s-1)$$

$$[(s-1)(s+1) - 3]Y = bs - b + a$$

$$(s^2-4)Y = bs - b + a$$

$$\boxed{Y = \frac{bs - b + a}{s^2-4} = \frac{bs + (a-b)}{(s+2)(s-2)}}$$

__Now__ You are asked to find values for a and b so X has only one pole. That means you must find a and b so $as + (a+3b)$ cancels either $s+2$ or $s-2$. There are infinitely many correct answers.

Continued on next page

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One correct answer is $a=1$ $b=-1$ So

$$X = \frac{as+(a+3b)}{(s+2)(s-2)} = \frac{\cancel{s-2}}{(s+2)\cancel{(s-2)}} = \frac{1}{s+2}$$

Only one pole (at $s=-2$)

You are also asked to check that Y will also have just one pole in this case.

$$Y = \frac{bs+(a-b)}{(s+2)(s-2)} = \frac{-s+2}{(s+2)(s-2)} = \frac{-\cancel{(s-2)}}{(s+2)\cancel{(s-2)}} = \frac{-1}{s+2}$$

Only one pole (at $s=-2$)

For information only, you aren't expected to include this in your answer)

Any pair a, b with $a=-b$ will work
OR $a=3b$

It is tricky to see why the same initial values reduce X and Y simultaneously using Laplace transforms. If you use matrix techniques it is much easier to see why this happens.

Geometrically, this system is a saddle since you have 2 poles, one negative, which leads to decay, and one positive, which leads to growth. This causes the saddle behavior where the solution oes in and then turns and heads out. Finding the initial conditions where there is only one pole is useful for finding where the solution curves go straight in or straight out without turning, which is a key to finding the ridges between where solutions turn one way or the other