Name:____________________________________

1. Solve the initial value problem $y''+4y'+3y=0$, $y(0) = 1$, $y'(0) = 0$.

2nd order
Linear
Constant Coeff
Homogeneous

$(D^2+4D+3)\,y=0$

$(D+3)(D+1)$

roots $-3, -1$

$y = C_1 e^{-3x} + C_2 e^{-x}$

$y' = -3C_1 e^{-3x} - C_2 e^{-x}$

$y(0) = C_1 + C_2 \overset{\text{set}}{=} 1$

$y'(0) = -3C_1 - C_2 \overset{\text{set}}{=} 0$

$-2C_1 = 1$

$C_1 = -\frac{1}{2}$

$C_2 = \frac{3}{2}$

$$\boxed{y = -\frac{1}{2}e^{-3x} + \frac{3}{2}e^{-x}}$$

Name:____________________________________

2. Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x^2 - 2xy + y^2}{x^2 - 2xy + 3y^2}$, $y(0) = 2$.

$$(x^2 - 2xy + 3y^2)\,dy = (x^2 - 2xy + y^2)\,dx$$

$$\underbrace{(-x^2 + 2xy - y^2)}\,dx + (x^2 - 2xy + 3y^2)\,dy = 0$$

$$\frac{\partial}{\partial y} = 2x - 2y \overset{\checkmark}{=} \frac{\partial}{\partial x} = 2x - 2y \qquad \boxed{\text{EXACT}}$$

$$F = \int -x^2 + 2xy - y^2 \, \partial x = -\frac{x^3}{3} + x^2y - xy^2 + C(y)$$

$$F = \int x^2 - 2xy + 3y^2 \, \partial y = x^2y - xy^2 + y^3 + \tilde{C}(x)$$

$$F = -\frac{x^3}{3} + x^2y - xy^2 + y^3 = K$$

Now $y(0) = 2$

$$-0 + 0 - 0 + 8 = K$$

$$\boxed{-\frac{x^3}{3} + x^2y - xy^2 + y^3 = 8}$$

Name:____

3. Find the general solution to $x^2 y'' + 6xy' + 4y = 0$.

Euler Equation

$y = x^r$

$r(r-1)x^r + 6rx^r + 4x^r = 0$

$r(r-1) + 6r + 4 = 0$

$r^2 + 5r + 4 = 0$

$(r+4)(r+1) = 0$

$r = -4$ OR $r = -1$

$$\boxed{y = C_1 x^{-4} + C_2 x^{-1}}$$

Name:____________________________________

4. Solve the initial value problem $y''+6y'+10y=\delta(t)$, $y(0) = 1$, $y'(0) = 0$.

2nd order
Linear
Constant Coeff.
δ-fct
- Laplace transform technique

Let
$\bar{Y} = \mathcal{L}\{y\}$

$$\mathcal{L}\{y''+6y'+10y\} = \mathcal{L}\{\delta\}$$

$$s^2\bar{Y} - s\overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} + 6(s\bar{Y} - \overset{1}{\cancel{y(0)}}) + 10\bar{Y} = 1$$

$$(s^2+6s+10)\bar{Y} - s - 6 = 1$$

$$\bar{Y} = \frac{s+7}{s^2+6s+10} = \frac{s+7}{(s+3)^2+1^2}$$

$$\frac{s+7}{(s+3)^2+1^2} = A\frac{s+3}{(s+3)^2+1^2} + B\frac{1}{(s+3)^2+1^2}$$

$$s+7 = A(s+3)+B = As+3A+B$$

s: $1 = A$

Const: $7 = 3A+B \rightarrow B = 4$

$$\boxed{y(t) = \left(e^{-3t}\cos(t) + 4e^{-3t}\sin(t)\right)u(t)}$$

Name:________________________________

5. Find all solutions to $y''+6y'+8y=40\cos(2x)$.

2nd order linear const. coeff. General solution so use Chap 2 (Undetermined Coeff)

$$(D^2+6D+8)y_h=0$$

$$(D+4)(D+2)$$

roots $-4, -2$

$$y_h = c_1e^{-4x}+c_2e^{-2x}$$

Real Methods

Guess $y_p = A\cos(2x)+B\sin(2x)$

$$y_p' = -2A\sin(2x)+2B\cos(2x)$$

$$y_p'' = -4A\cos(2x)-4B\sin(2x)$$

$$y_p''+6y_p'+8y_p$$

$$= (-4A+12B+8A)\cos(2x)$$

$$+(-4B-12A+8B)\sin(2x)$$

$$=(4A+12B)\cos(2x)+(-12A+4B)\sin(2x) \overset{\text{set}}{=} 40\cos(2x)$$

$$4A+12B=40 \rightarrow 12A+36B=120$$

$$-12A+4B=0 \qquad -12A+4B=0$$

$$40B=120$$

$$B=3$$

$$4A+12\cdot 3=40$$

$$A=1$$

$$y_p=\cos(2x)+3\sin 2x$$

OR

Complex Methods

$$y''+6y'+8y=\mathrm{Re}[40e^{i2x}]$$

Guess $\tilde{y}_p = Ae^{i2x}$

$$\tilde{y}_p' = 2iAe^{i2x}$$

$$\tilde{y}_p'' = -4Ae^{i2x}$$

$$\tilde{y}_p''+6\tilde{y}_p'+8\tilde{y}_p = (-4A+12iA+8A)e^{i2x}$$

$$=(4+12i)Ae^{i2x} \overset{\text{set}}{=} 40e^{i2x}$$

$$A=\frac{40}{4+12i}\cdot\frac{4-12i}{4-12i}=\frac{160-480i}{160}$$

$$=1-3i$$

$$y_p=\mathrm{Re}[\tilde{y}_p]=\mathrm{Re}[(1-3i)e^{i2x}]$$

$$=\mathrm{Re}[(1-3i)(\cos(2x)+i\sin(2x))]$$

$$=\cos(2x)+3\sin(2x)$$

So the general solution is

$$\boxed{y=\cos(2x)+3\sin(2x)+c_1e^{-4x}+c_2e^{-2x}}$$

Name:____________________________________

2nd order linear variable coeff. series

6. Solve the initial value problem $y''+(x+2)y'+2y=0$, $y(0) = 1$, $y'(0) = 2$.

Step1

$$y=\sum_{n=0}^{\infty} a_n x^n \qquad 2y=\sum_{n=0}^{\infty} 2a_n x^n$$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \qquad (x+2)y'=\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Step2

Convert to x^{index}: $\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{j=0}^{\infty} 2(j+1) a_{j+1} x^j$

(Let $j=n-1$, $n=j+1$ $n=1 \to j=0$)

(let $k=n-2$ $n=k+2$, $n=2 \to k=0$) $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$

Step3

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=1}^{\infty} m a_m x^m+\sum_{m=0}^{\infty} 2(m+1)a_{m+1}x^m+\sum_{m=0}^{\infty} 2a_m x^m=0$$

Step4

$$\underbrace{(2a_2+2a_1+2a_0)}_{m=0}+\sum_{m=1}^{\infty}\begin{bmatrix}(m+2)(m+1)a_{m+2}+ma_m \\ +2(m+1)a_{m+1}+2a_m\end{bmatrix}x^m=0$$

Step5

$$2a_2+2a_1+2a_0=0 \to a_2=-a_0-a_1$$

$(m\geq 1)$

$$(m+2)(m+1)a_{m+2}+ma_m+2(m+1)a_{m+1}+2a_m=0 \to a_{m+2}=-\frac{2(m+1)a_{m+1}+(m+2)a_m}{(m+2)(m+1)}$$

Step6 From initial conditions $a_0=1$
$a_1=2$
$a_2=-a_0-a_1=-3$

$$\boxed{y=1+2x-3x^2+x^3+\tfrac{1}{2}x^4+\dots}$$

$(m=1)$ $a_3=-\frac{2(2)(-3)+3(2)}{3\cdot 2}=1$

$a_4=-\frac{2(3)(1)+4(-3)}{4\cdot 3}=\frac{1}{2}$

⋮

Name:______________________________

7. Solve the boundary value problem $y''+4y=0$, $y(0) = 0$, $y(\pi)=1$.

2nd order
Linear
Const. Coeff
BOUNDARY VALUE

$(D^2+4)y=0$

roots $\pm 2i$

$y = C_1\cos(2x) + C_2\sin(2x)$

$y(0) = C_1\cancel{\cos(0)}^{1} + C_2\cancel{\sin(0)}^{0} = C_1 \underline{\underline{\text{Set}}}\ 0$

$y(\pi) = C_1\cos\cancel{(2\pi)}^{1} + C_2\cancel{\sin(2\pi)}^{0} = C_1 \underline{\underline{\text{Set}}}\ 1$

X

Can't have $C_1=0$ and 1

$\boxed{\text{NO SOLUTION}}$

Name:____________________________________

$\underline{\text{Difference}}$
$\underline{\text{Equation}}$

8. Solve the initial value problem $2a_{n+2} - 3a_{n+1} + a_n = 0$, $a_0 = 1$, $a_1 = 2$.

Guess $a_n = \lambda^n$

$$2\lambda^{n+2} - 3\lambda^{n+1} + \lambda^n = 0$$

$$2\lambda^2 - 3\lambda + 1 = 0$$

$$(2\lambda - 1)(\lambda - 1) = 0$$

roots $1/2$, 1

$$a_n = C_1\left(\tfrac{1}{2}\right)^n + C_2(1)^n$$

$$= \frac{C_1}{2^n} + C_2$$

$$a_0 = \frac{C_1}{2^0} + C_2 = C_1 + C_2 \overset{\text{Set}}{=} 1$$

$$a_1 = \frac{C_1}{2^1} + C_2 = \left(\tfrac{1}{2}C_1 + C_2 \overset{\text{Set}}{=} 2\right)$$

$$\tfrac{1}{2}C_1 = -1$$

$$C_1 = -2$$

$$C_2 = 3$$

$$\boxed{a_n = \frac{-2}{2^n} + 3 = 3 - 2^{1-n}}$$

Name:__________________________________

9. What is $y(1)$ if $y(x)$ is the solution to $\frac{dy}{dx} = 3y - 10e^x y^4$, $y(0) = 1/8$.

1st order

Bernoulli

$$\frac{dy}{dx} - 3y = -10e^x y^4$$

Let $y = v^{\frac{1}{1-4}} = v^{-\frac{1}{3}}$

$$\frac{dy}{dx} = -\frac{1}{3} v^{-4/3} \frac{dv}{dx}$$

Multiply by $-3v^{4/3}$

$$-\frac{1}{3} v^{-\frac{4}{3}} \frac{dv}{dx} - 3v^{-\frac{1}{3}} = -10e^x v^{-4/3}$$

This is 1st order linear constant coeff. Can be solved using paradigm from Chapter 1 or Chapter 2. I will use Chapter 2 techniques here

$$\frac{dv}{dx} + 9v = +30e^x$$

$$V_h = C_1 e^{-9x}$$

Guess $V_p = Ae^x$ $\quad V_p' + 9V_p = Ae^x + 9Ae^x$

$$= 10Ae^x \overset{\text{set}}{=} 30e^x$$

$$A = 3$$

$$V = C_1 e^{-9x} + 3e^x$$

So $\quad y = \left(C_1 e^{-9x} + 3e^x\right)^{-1/3}$

$$y(0) = \left(C_1 + 3\right)^{-1/3} = \frac{1}{8}$$

$$C_1 + 3 = 512$$

$$C_1 = 509$$

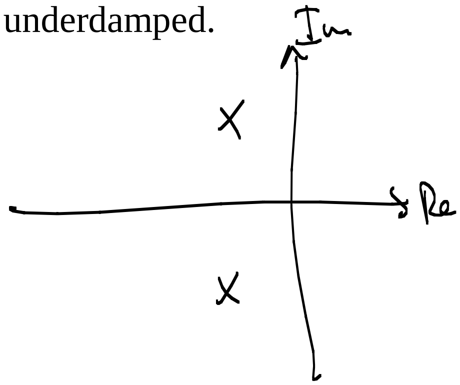
$$y = \left(509e^{-9x} + 3e^x\right)^{-1/3}$$

$$\boxed{y(1) = \left(509e^{-9} + 3e\right)^{-1/3}} \approx 0.4955\ldots$$

Name:____________________________________

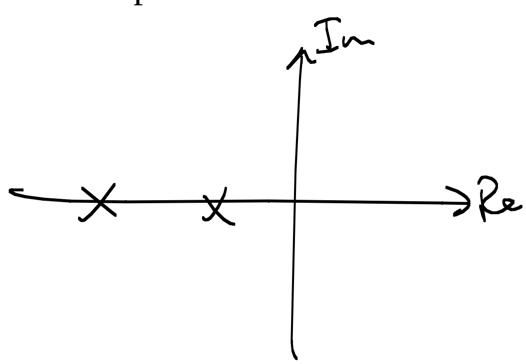
10. $m\dfrac{d^2x}{dt^2}+c\dfrac{dx}{dt}+kx=0$ is the equation for the motion of a spring-mass system. Graph a possible set of locations of the poles of the Laplace transform of x for each of the following situations.

(A) The spring is underdamped.



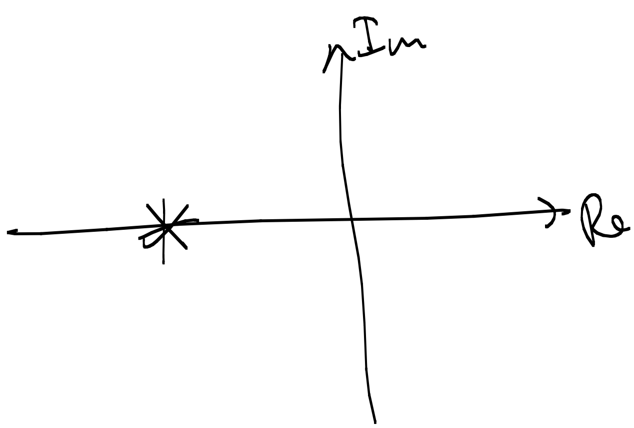
poles have negative real part and are complex conjugate (non-zero imaginary part)

(B) The spring is overdamped.



Distinct negative real poles

(C) The spring is critically damped.



double pole on the negative real line

Name:____________________________________

11. Approximate $y(.3)$ using Euler's method with step size 0.1 where
$y''-y'+xy=0, \quad y(0)=1, \ y'(0)=1.$

Write as a system $\frac{dy}{dx}=v$

$$\frac{dv}{dx}=\frac{d^2y}{dx^2}=y'-xy=v-xy$$

x	$\frac{dy}{dx}=$ old v	$\frac{dv}{dx}=$ old $v-xy$	y	v	$h=0.1$
0			1	1	
0.1	1	$1-0.1=1$	$1+0.1\cdot 1$ $=1.1$	$1+0.1\cdot 1$ $=1.1$	
0.2	1.1	$1.1-0.1\cdot 1.1$ $=0.99$	$1.1+0.1(1.1)$ $=1.21$	$1.1+0.1(.99)$ 1.199	
0.3	1.199	—	$1.21+0.1(1.199)$ $\boxed{1.3299}$	—	

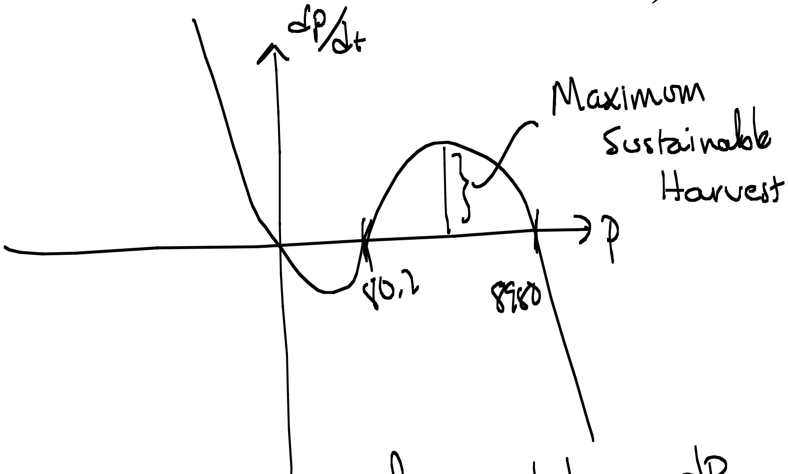
Name:__________________________________

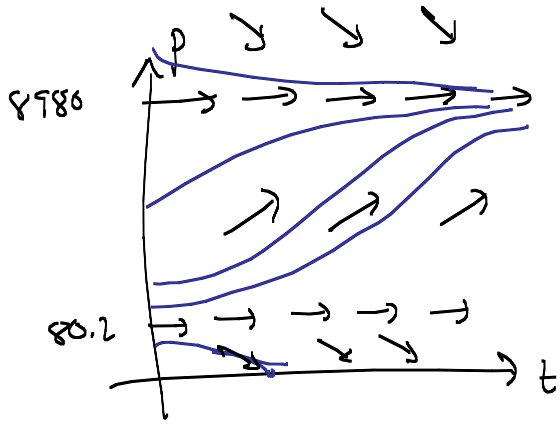
See HW 3 and lecture notes for 2/2

12. Suppose a population of fish is modeled by the equation

$$\frac{dp}{dt} = -1.00\times10^{-8}p^3 + 9.06\times10^{-5}p^2 - 7.20\times10^{-3}p$$

(This represents logistic growth with threshold with a carrying capacity of 8980 and a threshold of 80.2). What is the maximum amount you can harvest from this population without driving it to extinction (the maximum sustainable harvest)?





Harvesting changes the model to $\frac{dp}{dt} = -1.00\times10^{-8}p^3 + 9.06\times10^{-5}p^2 - 7.20\times10^{-3}p - h$

Amount Harvested

which shifts the dp/dt vs. p curve (on the left) down by h. This is sustainable as long as you keep a stable equilibrium, i.e. as long as the curve still gets above the p-axis. So the maximum sustainable harvest is the local max of dp/dt between the threshold (80.2) and the carrying capacity (8980)

To find local max of $-1\times10^{-8}p^3 + 9.06\times10^{-5}p^2 - 7.2\times10^{-3}p$, differentiate and set to 0.

$$-3\times10^{-8}p^2 + 1.812\times10^{-4}p - 7.2\times10^{-3} = 0$$

Quadratic Formula $p = \dfrac{-1.812\times10^{-4} \pm \sqrt{(-1.812\times10^{-4})^2 - 4(-3\times10^{-8})(-7.2\times10^{-3})}}{2(-3\times10^{-8})}$

$p = 40$ OR 6000 ← between 80.2 and 8980

At $p = 6000$, $\frac{dp}{dt} = -1\times10^{-8}\cdot6000^3 + 9.06\times10^{-5}\times6000^2 + 7.2\times10^{-3}\times6000$

$= \boxed{1058.4}$ Max Sustainable Harvest