Name:______________________________________

Recitation:______________________________________

Math 240
Exam 1
Sept. 23, 2014

Problem	**Score**
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Closed book. You may use a calculator and one 8 ½ × 11" sheet of handwritten notes (both sides). You must show your work to receive full credit. Please do all work on the test pages. The last page is intentionally left blank in case you need extra room.

Pledge:
On my honor, as a student, I have neither given nor received unauthorized aid on this

examination: ____ ____________

(signature) (date)

Name:______________________________________

1. Find all solutions to $\frac{dy}{dx} = 2x(y^2 - 1)$. Separable

$$\frac{dy}{y^2-1} = 2x\,dx$$

$$\int \frac{dy}{y^2-1} = \int 2x\,dx = x^2 + C$$

$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)}$$

$$\frac{1}{(y+1)(y-1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

$$1 = A(y+1) + B(y-1)$$

$y=1 \quad 1 = 2A \rightarrow A = 1/2$

$y=-1 \quad 1 = -2B \rightarrow B = -1/2$

$$\int \frac{1/2}{y-1} - \frac{1/2}{y+1}\,dy = x^2 + C$$

$$\frac{1}{2}\log|y-1| - \frac{1}{2}\log|y+1| = x^2 + C$$

$$\log\left|\frac{y-1}{y+1}\right| = 2x^2 + C$$

$(k = \pm e^c)$

$$\boxed{\frac{y-1}{y+1} = ke^{2x^2}}$$ General Solution

And singular solution $\boxed{y = -1}$

Sing. Soln.: Divided by y^2-1 which is 0 at $y = \pm 1$

$y = 1$ corresponds to $k = 0$

$y = -1$ singular

Note: You can rewrite $\frac{y-1}{y+1} = ke^{2x^2} \rightarrow y - 1 = ke^{2x^2}y + ke^{2x^2}$

$$y - ke^{2x^2}y = 1 + ke^{2x^2}$$

$$\boxed{y = \frac{1+ke^{2x^2}}{1-ke^{2x^2}}}$$ Explicit form of the general solution

Name:____________________________________

2. Find all solutions to $\frac{dy}{dx} = xy^2 - 2y$.

$$\frac{dy}{dx} + 2y = xy^2$$

Bernoulli $y = v^{\frac{1}{1-2}} = v^{-1}$

$$-v^{-2}\frac{dv}{dx} + 2v^{-1} = xv^{-2}$$

$$\frac{dy}{dx} = \frac{d\,v^{-1}}{dx} = -v^{-2}\frac{dv}{dx}$$

$$\frac{dv}{dx} - 2v = -x \quad \text{(Linear)}$$

$$\mu = e^{\int -2\,dx} = e^{-2x + \cancel{C}^{\,0}}$$ (only need 1 integrating factor)

$$\underbrace{e^{-2x}\frac{dv}{dx} - 2e^{-2x}v}_{\frac{d}{dx}\left(e^{-2x}v\right)} = -xe^{-2x}$$

$$\frac{d}{dx}\left(e^{-2x}v\right) = -xe^{-2x}$$

Integrate by parts
$u = x \quad dw = e^{-2x}$
$du = dx \quad w = -\frac{1}{2}e^{-2x}$

$$e^{-2x}v = -\int xe^{-2x}\,dx$$

$$e^{-2x}v = \frac{1}{2}xe^{-2x} - \frac{1}{2}\int e^{-2x}\,dx$$

$$e^{-2x}v = \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} + C$$

$$v = \frac{1}{2}x + \frac{1}{4} + Ce^{2x}$$

$$y = v^{-1} \longrightarrow \boxed{y = \frac{1}{\frac{1}{2}x + \frac{1}{4} + Ce^{2x}}}$$ General Solution

Check for sing. soln.
$y = 0$ also works $\longrightarrow$ AND $\boxed{y = 0}$ singular solution

Name:________________________________

3. Solve the initial value problem $\frac{dy}{dx} = \frac{3x^2 - 2y}{2x+5}$, $y(1) = 1$.

$$(2x+5)\,dy = (3x^2-2y)\,dx$$

$$\underbrace{(2x+5)}\,dy + (2y-3x^2)\,dx = 0$$

EXACT

$$\frac{\partial}{\partial x} = 2 \checkmark = \frac{\partial}{\partial y}$$

$$F = \int 2x+5\,\partial y = 2xy + 5y + \cancel{C(x)} \checkmark$$

$$F = \int 2y - 3x^2\,\partial x = 2xy - x^3 + \tilde{C}(y)$$

$$F = 2xy + 5y - x^3 = K \quad \text{General Solution}$$

Initial Value $y(1) = 1$

$$2\cdot 1\cdot 1 + 5\cdot 1 - 1^3 = K$$

$$6 = K$$

Solution to initial value problem $\boxed{2xy + 5y - x^3 = 6}$

(Note you can simplify to $\boxed{y = \frac{x^3+6}{2x+5}}$ but this is not required)

Name:____________________________________

4. Solve the initial value problem, $\frac{dy}{dx} = \cos(x) - 3x^2 y, \quad y(0) = 1$. Your answer will include an integral that you can't simplify.

$$\frac{dy}{dx} + 3x^2 y = \cos(x)$$

LINEAR

only need any one integrating factor

$$\mu = e^{\int 3x^2 dx} = e^{x^3 + \cancel{C}} = e^{x^3}$$

$$\underbrace{e^{x^3}\frac{dy}{dx} + e^{x^3}3x^2 y}_{\frac{d}{dx}\left(e^{x^3}y\right)} = e^{x^3}\cos(x)$$

$$\frac{d}{dx}\left(e^{x^3}y\right) = e^{x^3}\cos(x)$$

$$e^{x^3}y = \int e^{x^3}\cos(x)\,dx$$

$$= \int_0^x e^{t^3}\cos(t)\,dt + C$$

$$y = e^{-x^3}\int_0^x e^{t^3}\cos(t)\,dt + Ce^{-x^3}$$

General Solution

Now plug in $y(0) = 1$ to find C

$$1 = e^{-0^3}\int_0^0 e^{t^3}\cos(t)\,dt + Ce^{-0^3}$$

0 since integral from 0 to 0.

$$1 = C$$

$$\boxed{y = e^{-x^3}\int_0^x e^{t^3}\cos(t)\,dt + e^{-x^3}}$$

Name:____________________________________

5. Solve the initial value problem, $\frac{dy}{dx} = \frac{x}{2y} - \frac{y}{2x}$, $y(1) = 1/3$.

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

Put over common denominator

SEE LAST PAGE TO SEE HOW PROBLEM IS SOLVED AS HOMOGENEOUS

$$2xy\,dy = (x^2 - y^2)\,dx$$

$$\underbrace{(y^2 - x^2)}\,dx + \underbrace{2xy}\,dy = 0$$

EXACT

$$\frac{\partial}{\partial y} = 2y = \frac{\partial}{\partial x} \quad \checkmark$$

$$F = \int y^2 - x^2\,\partial x = xy^2 - \frac{x^3}{3} + C(y), \quad C(y) = 0$$

$$F = \int 2xy\,\partial y = xy^2 + \tilde{C}(x)$$

General Solution

$$xy^2 - \frac{x^3}{3} = K \quad \text{OR} \quad \underline{3xy^2 - x^3 = K}$$

(Get rid of fractions since 3 times an arbitrary constant is an arbitrary constant.)

Initial Value

$y(1) = 1/3$

$$3 \cdot 1 \left(\tfrac{1}{3}\right)^2 - 1^3 = K$$

$$\tfrac{1}{3} - 1 = K$$

$$-\tfrac{2}{3} = K$$

$$\boxed{3xy^2 - x^3 = -\tfrac{2}{3}}$$

(OR if you prefer $3x^3 - 9xy^2 = 2$)

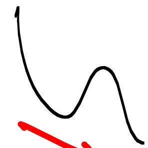
Name:____________________________________

6. Suppose $P(t)$ is the solution to the initial value problem

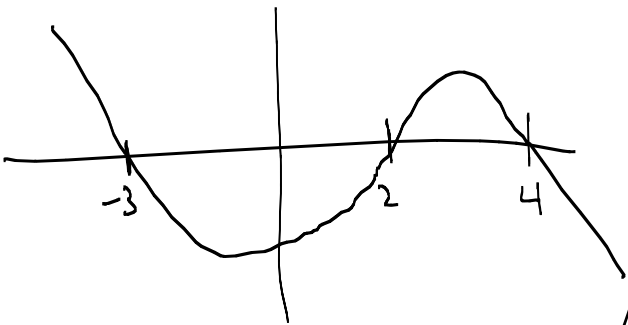
$$\frac{dP}{dt} = -(P-4)(P-2)(P+3),\ P(0) = P_0.$$

a) For what values of P_0 is $\lim_{t\to\infty} P(t)$ defined and positive?

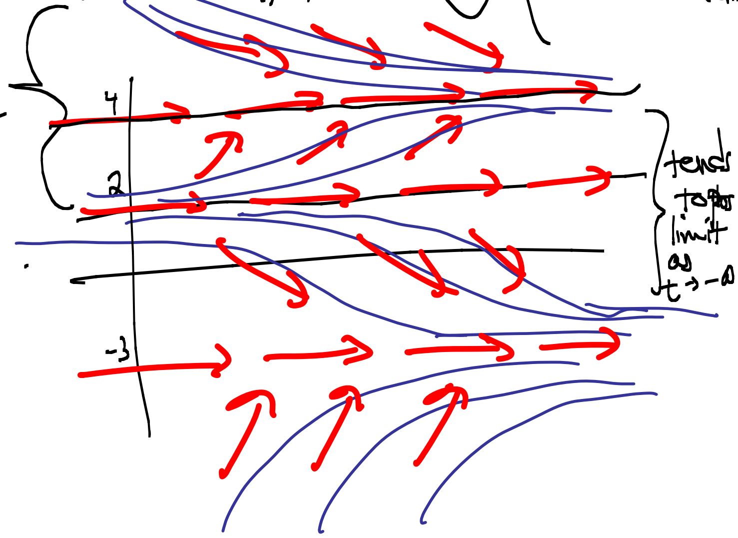
cubic – roots at 4, 2, −3



since $-P^3$ term



tends to pos. limit



tends to the limit as $t \to -\infty$

$$\boxed{\lim_{t\to\infty} P(t) > 0 \quad \text{for } P_0 \geq 2}$$

b) For what values of P_0 is $\lim_{t\to-\infty} P(t)$ defined and positive?

From the picture above $\lim_{t\to-\infty} P(t)$ is defined and positive if $-3 < P_0 \leq 4$

Note that $\lim_{t\to-\infty} P(t)$ is not defined if $P_0 > 4$, instead it explodes to ∞ in finite time since the polynomial $-(P+4)(P+2)(P-3)$ is of degree $3 > 1$.

This is a fine distinction and only worth 1 point

Name:________________________________

7. Using the improved Euler method with step size h = 1/2, approximate $y(1/2)$ if $\frac{dy}{dx} = 2x - y$, $y(0) = 1$.

x	left y'	$\tilde{y}$	right y'	y	h
0				1	$\frac{1}{2}$
$0 + \frac{1}{2} = \frac{1}{2}$	$2 \cdot 0 - 1 = -1$	$1 + \frac{1}{2}(-1) = \frac{1}{2}$	$2 \cdot \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$	$1 + \frac{1}{2}\left(\frac{-1 + \frac{1}{2}}{2}\right) = \boxed{\frac{7}{8}}$ (0.875)	

Name:____________________________________

8. Find values for a and b so that the solution to the initial value problem $\frac{dy}{dx} = x^2 - y^2 + a$, $y(-4) = b$, is a straight line.

Suppose the solution is a straight line. Then we can write the solution as $y = mx + k$ (using k since we already have a b)

Plugging this in we get

$$\frac{dy}{dx} = m \qquad x^2 - y^2 + a = x^2 - (mx+k)^2 + a$$
$$= x^2 - m^2x^2 - 2mkx - k^2 + a$$
$$= (1-m^2)x^2 - (2mk)x + (a-k^2)$$

So we need

$$m = (1-m^2)x^2 - (2mk)x + (a-k^2) \quad \underline{\text{for every } x}$$

This can only happen if the coefficients are the same on both sides

	Left		Right	
Coeff of x^2:	0	$=$	$1-m^2$	$\longrightarrow m = \pm 1$
Coeff of x:	0	$=$	$-2mk$	$\longrightarrow k = 0$ (since $m \neq 0$)
Constant:	m	$=$	$a - \cancel{k^2}^{\,0} = a$	

So $m = 1$, $k = 0$ works with $a = 1 \longrightarrow y = x$ solves $\frac{dy}{dx} = x^2 - y^2 + 1$

Then $y(-4) = -4$

So $\boxed{a = 1 \quad b = -4}$ ← one solution

There are two solutions but you only needed to find one

You could also find $m = -1$, $k = 0$ works with $a = -1 \rightarrow y = -x$ solves $\frac{dy}{dx} = x^2 - y^2 - 1$

Then $y(-4) = 4$

So $\underline{a = -1 \quad b = 4}$ is another solution.

Name:________________________________

Alternate Solution to 5

$$\frac{dy}{dx} = \frac{1}{2}\frac{x}{y} - \frac{1}{2}\frac{y}{x}$$ Homogeneous $\quad v = y/x$, $y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{1}{2v} - \frac{v}{2}$$

Now Separable

$$x\frac{dv}{dx} = \frac{1}{2v} - \frac{3v}{2} = \frac{1-3v^2}{2v}$$

$$\frac{2v}{1-3v^2}\,dv = \frac{dx}{x}$$

$u = 1-3v^2$
$du = -6v\,dv$

$$\int \frac{2v}{1-3v^2}\,dv = \int\frac{dx}{x} = \log|x| + C$$

$$-\frac{1}{3}\int\frac{du}{u} = \log|x| + C$$

$$-\frac{1}{3}\log|1-3v^2| = \log|x| + C$$

$$\log|1-3v^2| = -3\log|x| + C$$

$$1-3v^2 = kx^{-3}$$

$$1 - 3\frac{y^2}{x^2} = \frac{k}{x^3}$$

$$x^3 - 3xy^2 = k$$

Plug in $y(1) = 1/3$

$$1 - 3\cdot 1\cdot(1/3)^2 = k$$

$$2/3 = k$$

$$\boxed{x^3 - 3xy^2 = 2/3}$$

(or if you prefer $3x^3 - 9xy^2 = 2$)