Name:____

1. Solve the initial value problem $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$, $y(0) = 2$, $y'(0) = 4$.

$$(D^2+6D+13)y=0$$

$$\frac{-6\pm\sqrt{36-4\cdot 13}}{2} = \frac{-6\pm\sqrt{-16}}{2} = -3\pm 2i$$

$$y = c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x)$$

$$y' = -3c_1 e^{-3x}\cos(2x) - 2c_1 e^{-3x}\sin(2x) + 2c_2 e^{-3x}\cos(2x)$$
$$\qquad - 3c_2 e^{-3x}\sin(2x)$$

$$y(0) = c_1\cdot 1 + c_2\cdot 0 = c_1 \overset{\text{set}}{=} 2$$

$$y'(0) = -3c_1\cdot 1 - 2c_1\cdot 0 - 3c_2\cdot 0 + 2c_2\cdot 1 = -3c_1 + 2c_2 \overset{\text{set}}{=} 4$$

$$c_1 = 2 \qquad -3\cdot 2 + 2c_2 = 4$$
$$2c_2 = 10$$
$$c_2 = 5$$

$$\boxed{y = 2e^{-3x}\cos(2x) + 5e^{-2x}\sin(2x)}$$

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2. Find a particular solution to $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \exp(2x)$.

Check Form of Homo. Soln.

$(D^2 + 4D + 5)$

$\frac{-4 \pm \sqrt{16-20}}{2}$

$-2 \pm i$

So e^{2x} is NOT part of the homogeneous sol'n.

Guess $y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$y_p'' + 4y_p' + 5y_p = (4A + 8A + 5A)e^{2x} \overset{\text{Want}}{=} e^{2x}$

$17A e^{2x} = e^{2x}$

$A = 1/17$

$\boxed{y_p = \frac{1}{17}e^{2x}}$

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3. An undamped spring-mass system with a mass of 4kg is observed to have a natural frequency of 2 cycles per second. What is the spring constant?

$$2 \, \cancel{\text{cycles}}/\text{sec} \times 2\pi \text{ radians}/\cancel{\text{cycle}} = 4\pi \text{ radians}/\text{sec}$$

$$4\pi \frac{1}{\text{sec}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{4\text{kg}}}$$

$$16\pi^2 \frac{1}{\text{sec}^2} = \frac{k}{4\text{kg}}$$

$$\boxed{64\pi^2 \frac{\text{kg}}{\text{sec}^2} = k} \quad (64\pi^2 = 631.65\ldots)$$

Note: Radian measure is defined as $\frac{\text{Arc length}}{\text{Radial length}}$ so is dimensionless

Note: $\frac{\text{kg}}{\text{sec}^2} = \frac{\frac{\text{kg m}}{\text{sec}^2}}{\text{m}} = \frac{\text{Joules}}{\text{meter}}$

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4. Rewrite $12\cos(9x)+5\sin(9x)$ in the form $A\cos(9x+\phi)$.

$$12\cos(9x)+5\sin(9x) = \mathrm{Re}\left[(a+bi)(\cos(9x)+i\sin(9x))\right] = \mathrm{Re}\left[(a+bi)e^{i9x}\right]$$

$$= \mathrm{Re}\left[a\cos 9x \underset{\uparrow\ \text{From } i^2}{-} b\sin 9x + i\underset{\uparrow}{(\cdots)}\right]$$

irrelevant since looking for real part

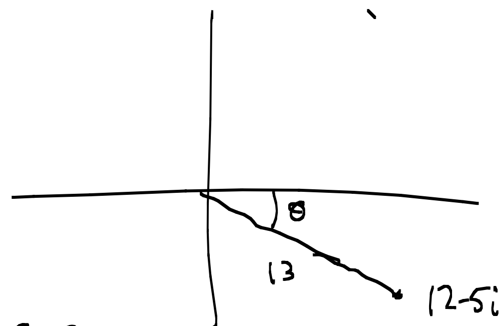
So $a = 12$
$b = -5$

$$12\cos(9x)+5\sin(9x) = \mathrm{Re}\left[(12-5i)e^{i9x}\right]$$

Now write $12-5i$ in polar form

$$|12-5i| = \sqrt{12^2+5^2} = 13$$

$$\theta = \operatorname{atan}\left(\frac{-5}{12}\right) \quad \left(\begin{gathered} = -\operatorname{atan}(5/12) \\ \text{OR} \\ = -.39479\ldots \end{gathered}\right)$$



In 4th quadrant so arctan works

$$\mathrm{Re}\left[(12-5i)e^{i9x}\right] = \mathrm{Re}\left[13e^{-\operatorname{atan}(5/12)}e^{i9x}\right]$$

$$= \mathrm{Re}\left[13e^{i(9x-\operatorname{atan}(5/12))}\right]$$

$$= \boxed{13\cos 9x - \operatorname{atan}(5/12)}$$

Note: the usual calculus rules, such as $\frac{d}{dx}\sin(x) = \cos(x)$, only hold for radian measure, so in this class as in all calculus classes you must use radians

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5. Using the improved Euler's method with step size $h = 0.1$, approximate $y(0.2)$ if $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 20y = 0, \quad y(0) = 1, \quad y'(0) = 0$.

(**Note**: It is probably best if you *don't* try to check your work by finding the exact answer. The exact solution is very messy and this step-size is too large for this approximation to be very accurate, so you won't actually be close to the correct answer).

Let $\dfrac{dy}{dx} = v \qquad \dfrac{dv}{dx} = 10v - 20y$

$y(0) = 1 \qquad v(0) = 0$

x	left y'	left v'	$\tilde{y}$	$\tilde{v}$	right y'	right v'	y	v
0	—	—	—	—	—	—	1	0
0.1	0	-20	1	$0 + .1 \times (-20) = -2$	-2	$10(-2) - 20(1) = -40$	$1 + .1\left(\frac{0 + -2}{2}\right) = 0.9$	$0 + .1\left(\frac{-20 + -40}{2}\right) = -3$
0.2	-3	$10(-3) - 20(.9) = -48$	$.9 + .1(-3) = 0.6$	$-3 + .1(-48) = -7.8$	-7.8	Not needed	$.9 + .1\left(\frac{-3 + -7.8}{2}\right) = 0.36$	Not needed

$$\boxed{y(0.2) \approx 0.36}$$

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6. Match the following differential equations with the graphs of their solutions.

(a) $x''+4x=0, \quad x(0)=0,\ x'(0)=8$

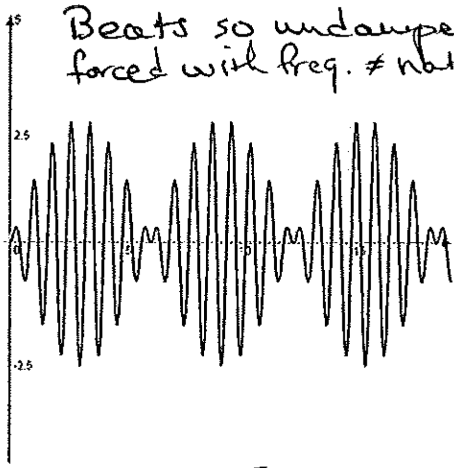
(b) $x''+x'+4x=0, \quad x(0)=0,\ x'(0)=8$

(c) $x''+5x'+4x=0, \quad x(0)=0,\ x'(0)=8$

(d) $x''+49x=3\cos(7t), \quad x(0)=0,\ x'(0)=0$

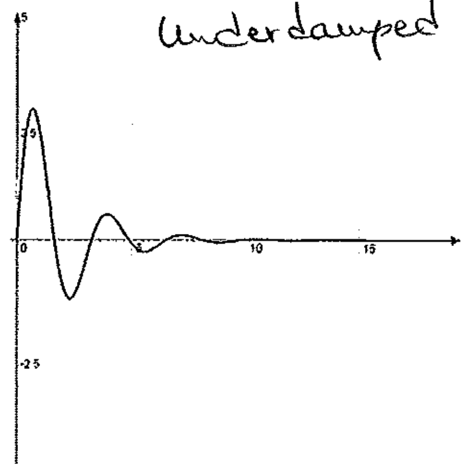
(e) $x''+49x=20\cos(8t), \quad x(0)=0,\ x'(0)=0$

Beats so undamped forced with freq. ≠ natural freq.



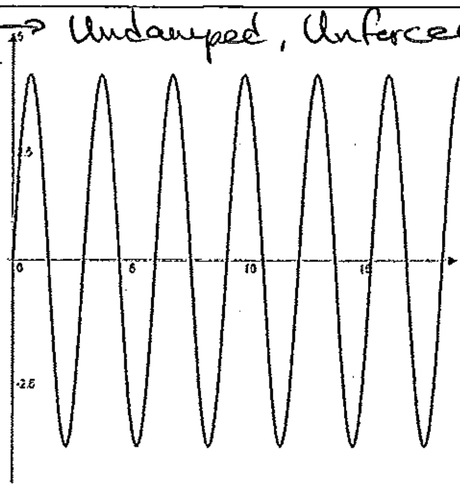
Equation E

Underdamped



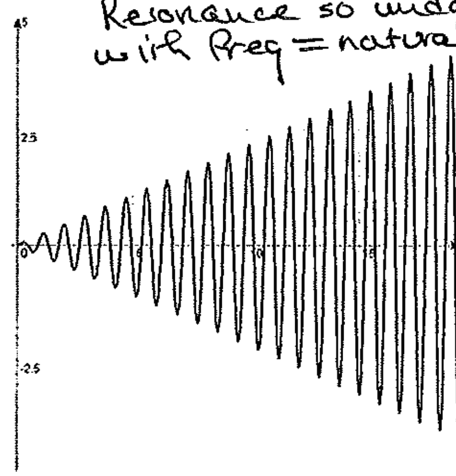
Equation B

Simple Harmonic Motion → Undamped, Unforced



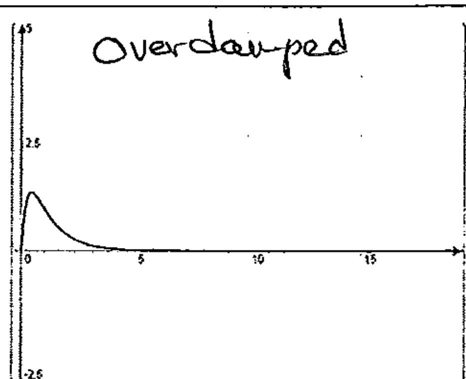
Equation A

Resonance so undamped, forced with freq = natural frequency



Equation D

Overdamped



Equation C

Name:____________________________________

7. Find the general solution to $\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}+12y=\dfrac{1}{x^2+1}$. Your answer will involve integrals which you can't simplify.

Homo Soln. $(D^2+8D+12)y=0$

$(D+6)(D+2)$

roots -6, -2

$y_h = c_1 e^{-6x} + c_2 e^{-2x}$

Since $\frac{1}{x^2+1}$ isn't a form for which we know how to guess, use Variation of Parameters

$y_1 = e^{-6x}$

$y_2 = e^{-2x}$

$$\begin{aligned} W(y_1,y_2) &= y_1 y_2' - y_1' y_2 \\ &= e^{-6x}(-2e^{-2x}) - (-6e^{-6x})e^{-2x} \\ &= -2e^{-8x} + 6e^{-8x} = 4e^{-8x} \end{aligned}$$

$$y = -e^{-6x}\int^x \frac{e^{-2t}\frac{1}{t^2+1}}{4e^{-8t}}\,dt + e^{-2x}\int^x \frac{e^{-6t}\frac{1}{t^2+1}}{4e^{-8t}}\,dt + c_1 e^{-6x} + c_2 e^{-2x}$$

$$= -\frac{e^{-6x}}{4}\int^x \frac{e^{6t}}{t^2+1}\,dt + \frac{e^{-2x}}{4}\int^x \frac{e^{2t}}{t^2+1}\,dt + c_1 e^{-6x} + c_2 e^{-2x}$$

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8.

(a) Give the definition of a linear operator.

An operator takes functions to functions. It is linear if both

(i) $L(y+z) = L(y) + L(z)$

and (ii) $L(cy) = cL(y)$

for all functions y, z and constants c

(b) Is $Ly = \dfrac{d^2 y}{dx^2} + xy$ a linear operator? Justify your answer.

We check (i) $L(y+z) = \dfrac{d^2(y+z)}{dx^2} + x(y+z) = \dfrac{d^2 y}{dx^2} + \dfrac{d^2 z}{dx^2} + xy + xz$

$= \dfrac{d^2 y}{dx^2} + xy + \dfrac{d^2 z}{dx^2} + xz$

$= Ly + Lz$ ✓

and (ii) $L(cy) = \dfrac{d^2(cy)}{dx^2} + x(cy) = c\dfrac{d^2 y}{dx^2} + cxy$

$= c\left(\dfrac{d^2 y}{dx^2} + xy\right) = cLy$ ✓

So L is a linear operator