Name: Key

Recitation:

Math 240
Exam 3
November 18, 2014

Problem	**Score**
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Closed book. You may use a calculator and one 8 ½ × 11" sheet of handwritten notes (both sides). You must show your work to receive full credit. Write solutions in explicit form if possible. All problems have a solution that can be found using the techniques of this class. **Series solutions should be listed at least through the x^4 term unless otherwise specified.**

Pledge:
On my honor, as a student, I have neither given nor received unauthorized aid on this examination: ____ ____________

(signature) (date)

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1. Note: you may use the table of Laplace transforms which is attached at the end of the test.

a) Find the Laplace transform of $f(t) = 2\cos(t) + 1$

$$\mathcal{L}\{2\cos(t)+1\} = 2\mathcal{L}\{\cos(t)\} + \mathcal{L}\{1\}$$

$$= 2\frac{s}{s^2+1} + \frac{1}{s} \quad \text{(from table)}$$

$$= \frac{2s^2}{s(s^2+1)} + \frac{s^2+1}{s(s^2+1)}$$

$$= \boxed{\frac{3s^2+1}{s^3+s}}$$

b) Find the inverse Laplace transform of $F(s) = \dfrac{2s+2}{s^2+6s+13}$.

$e^{-3t}\cos(2t)$ $\quad$ $e^{-3t}\sin(2t)$

$$\frac{2s+2}{s^2+6s+13} = \frac{2s+2}{(s^2+6s+9)+4} = \frac{2s+2}{(s+3)^2+2^2} = A\frac{s+3}{(s+3)^2+2^2} + B\frac{2}{(s+3)^2+2^2}$$

Multiplying by $(s+3)^2+2^2$ $\rightarrow$

$$2s+2 = A(s+3) + 2B$$
$$2s+2 = As + (3A+3B)$$

s: $2 = A$

Const: $2 = 3A+2B \rightarrow 2 = 6+2B \rightarrow B = -2$

$$\boxed{f(t) = 2e^{-3t}\cos(2t) - 2e^{-3t}\sin(2t)}$$

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2. Solve the initial value problem

$$y'' + 8y' + 15y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

$$\left(s^2 Y - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0}\right) + 8\left(sY - \cancel{y(0)}^{0}\right) + 15Y = 1$$

$$(s^2 + 8s + 15)Y = 1$$

$$Y = \frac{1}{s^2 + 8s + 15}$$

$$\frac{1}{s^2 + 8s + 15} = \frac{1}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$1 = A(s+5) + B(s+3)$$

$s = -3$ $\quad 1 = 2A \rightarrow A = \frac{1}{2}$

$s = -5$ $\quad 1 = -2B \rightarrow B = -\frac{1}{2}$

$$\boxed{y(t) = \left(\frac{1}{2}e^{-3t} - \frac{1}{2}e^{-5t}\right)u(t)}$$

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3. Solve the initial value problem,

$$y''+(x+4)y=0, \quad y(0)=1, \; y'(0)=-4.$$

Variable Coeff ⇒ Series Soln

Step1 $y=\sum_{n=0}^{\infty} a_n x^n \qquad (x+4)y=\sum_{n=0}^{\infty} a_n x^{n+1}+\sum_{n=0}^{\infty} 4a_n x^n$

$y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

Step2 Let $j=n+1$

$\sum_{n=0}^{\infty} a_n x^{n+1}=\sum_{j=1}^{\infty} a_{j-1}x^j$

$\sum_{n=0}^{\infty} 4a_n x^n$ already in desired form

Let $k=n-2$

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$

Step3

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=1}^{\infty} a_{m-1}x^m+\sum_{m=0}^{\infty} 4a_m x^m=0$$

Step4 $(m=0)$

$$2a_2+4a_0+\sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2}+a_{m-1}+4a_m\right]x^m=0$$

(Recurrence Relation)

Step5 $2a_2+4a_0=0 \rightarrow a_2=-2a_0$

$(m\geq 1)$ $(m+2)(m+1)a_{m+2}+a_{m-1}+4a_m=0 \Rightarrow a_{m+2}=-\dfrac{4a_m+a_{m-1}}{(m+2)(m+1)}$

Step6 We know $a_0=y(0)=1$

$a_1=y'(0)=-4$

Then $a_2=-2a_0=-2$

$(m=1)$ $a_3=-\dfrac{4a_1+a_0}{3\cdot 2}=-\dfrac{-15}{6}=\dfrac{5}{2}$

$(m=2)$ $a_4=-\dfrac{4a_2+a_1}{4\cdot 3}=-\dfrac{-8-4}{12}=1$

$\vdots$

$$\boxed{y(x)=1-4x-2x^2+\frac{5}{2}x^3+x^4+\cdots}$$

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4. Solve the initial value problem (your solution will involve an integral)

$$y''+4y'+5y=f(t),\quad y(0)=0,\quad y'(0)=0$$

While you can use variation of parameters, convolution is easier

$$\left(s^2 Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right) + 4\left(sY - \cancel{y(0)}^{0}\right) + 5Y = F$$

$$(s^2+4s+5)Y = F$$

$$Y = \frac{1}{s^2+4s+5} F$$

$$\frac{1}{s^2+4s+5} = \frac{1}{s^2+4s+4+1} = \frac{1}{(s+2)^2+1^2}$$

So $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = e^{-2t}\sin(t)$

$\mathcal{L}^{-1}\{F\} = f(t)$

So $$y(t) = \int_0^t e^{-2T}\sin(T) f(t-T)\,dT$$

(or alternatively $\int_0^t e^{-2(t-T)}\sin(t-T) f(T)\,dT$

which works out to be the same)

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5. Find and classify the equilibria for the system,

$$\frac{dx}{dt} = 3x^2 + 4y^2 - 16,$$
$$\frac{dy}{dt} = x - 2y$$

Find equilibria

$3x^2+4y^2-16=0 \longrightarrow 3(2y)^2+4y^2-16=0$

$x-2y=0 \longrightarrow x=2y$

$3(4y^2)+4y^2=16$

$16y^2=16$

$y=\pm 1$

Equilibria are $(2,1)$ and $(2,-1)$

Classify

$\frac{dx}{dt}=f(x,y)$ $\quad \frac{\partial f}{\partial x}=6x \quad \frac{\partial f}{\partial y}=8y$

$\frac{dy}{dt}=g(x,y)$ $\quad \frac{\partial g}{\partial x}=1 \quad \frac{\partial g}{\partial y}=-2$

At $(2,1)$ $\quad \begin{pmatrix} 12 & 8 \\ 1 & -2 \end{pmatrix}$ $\quad$ $tr=10$, $det=-32<0$ $\Longrightarrow$ $\boxed{(2,1) \text{ is a SADDLE}}$

$tr^2-4\cdot det=228$

At $(-2,-1)$ $\quad \begin{pmatrix} -12 & -8 \\ 1 & -2 \end{pmatrix}$ $\quad$ $tr=-14<0$, $det=32>0$ $\Longrightarrow$ $\boxed{(-2,-1) \text{ converges straight in (nodal sink)}}$

$tr^2-4\cdot det=196-128=68>0$

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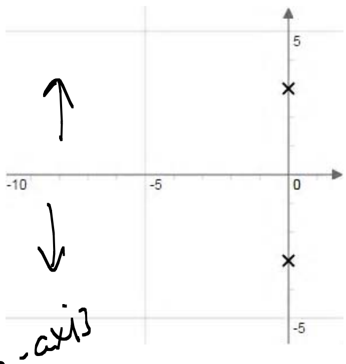
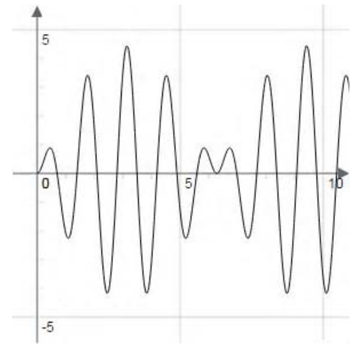
6. Match the poles of the Laplace transform on the left with the graphs of the functions on the right.

frequency larger as you get further from Re-axis

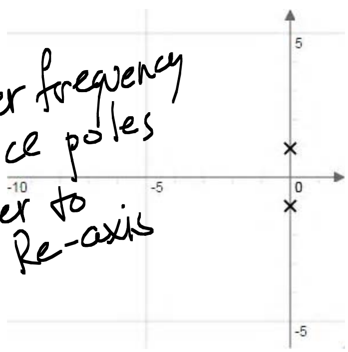


C

(A)

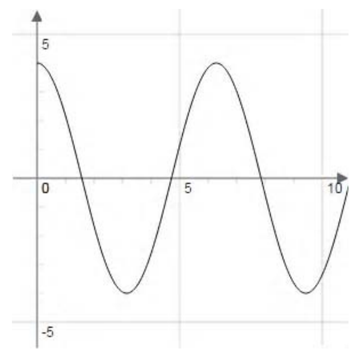


Smaller frequency since poles closer to Re-axis

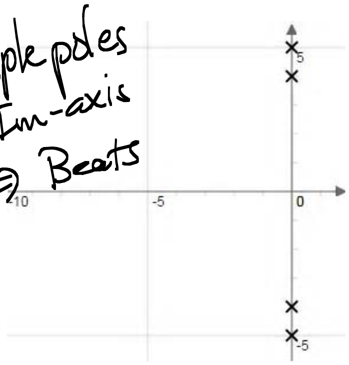


B

(B)

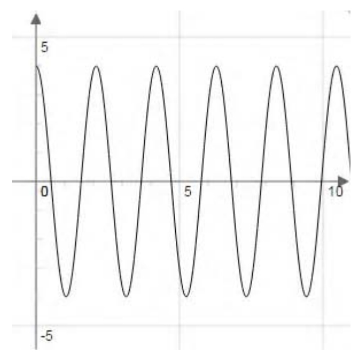


Multiple poles on Im-axis $\Rightarrow$ Beats

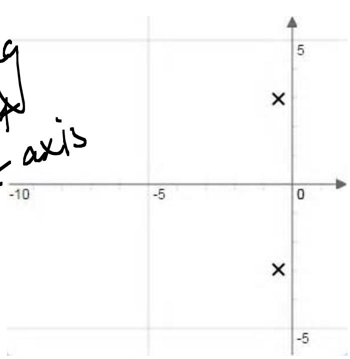


A

(C)

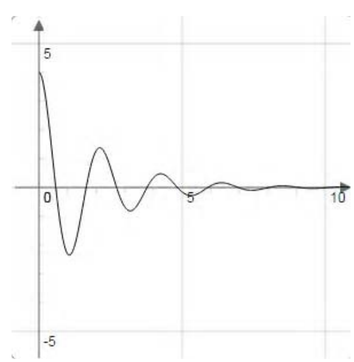


Decaying since left of Im-axis



D

(D)



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7. The recurrence relation for the series solution $y=\sum_{n=0}^{\infty} a_n x^n$ for the equation $y''-xy'+2y=0$ is $a_{n+2}=\dfrac{n-2}{(n+2)(n+1)}a_n$ for $n \geq 0$. If you are given $y''(0)=4$, what is the value of $y(0)$?

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$y''(x) = 2a_2 + 6a_3 x + \cdots$$

So $y''(0) = 2a_2$ (You could just remember this from homework)

So given $y''(0) = 4$
we have $2a_2 = 4$
$a_2 = 2$

Then by the recurrence relation

(n=0) $a_2 = \dfrac{0-2}{(0+2)(0+1)} a_0$

$2 = -\dfrac{2}{2} a_0$

$2 = -a_0$

$a_0 = -2$

So $\boxed{y(0) = -2}$

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8. Suppose x and y satisfy the system $\begin{aligned} \frac{dx}{dt} &= 10x - 2xy, \quad & x(0) = 4 \\ \frac{dy}{dt} &= 3y - xy, \quad & y(0) = 4 \end{aligned}$.

Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

This is an example of a competing species model that we analyzed in problems 38, 39, 40 of the homework for chapter 3

$\frac{dx}{dt} = 0$ when $10x - 2xy = 0$

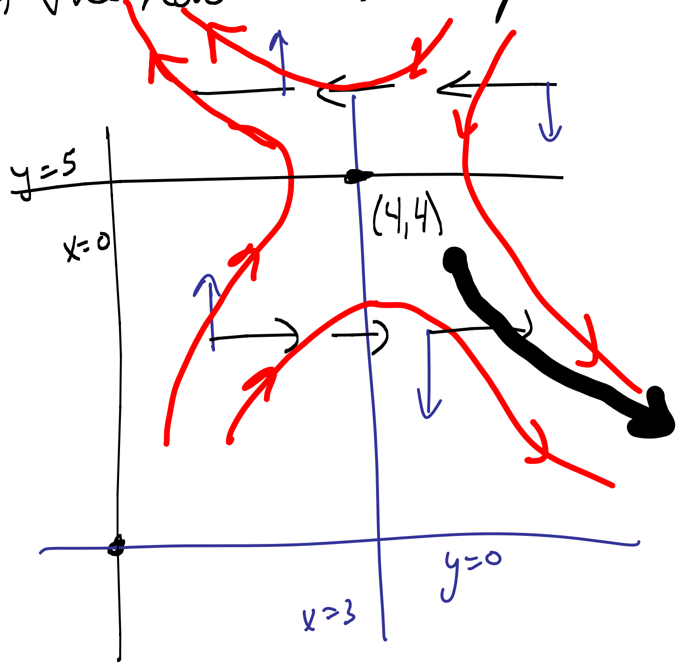
$(10-2y)x = 0$

$y = 5$ OR $x = 0$

$\frac{dy}{dt} = 0$ when $3y - xy = 0$

$(3-x)y = 0$

$x = 3$ OR $y = 0$



Equilibria at $(0,0)$ and $(3,5)$

You can classify $(0,0)$ as source (straight out) and $(3,5)$ as saddle as you did on the homework, but that is NOT necessary to solve the problem.

If we draw the arrows to show how the solutions move we get the diagram above, where the red lines show typical paths. We are given we start at $(4,4)$, so we follow the heavy black curve.

From this we can see

$$\boxed{\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0}$$